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A Mass Minimizing Flow for Real-Valued Flat Chains with Applications to Transport Networks

by

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Abstract

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An oriented transportation network can be modeled by a 1-dimensional chain whose boundary is the difference between the demand and supply distributions, represented by weighted sums of point masses. To accommodate efficiencies of scale into the model, one uses a suitable $\mathbf{M}^\alpha$ norm for transportation cost. One then finds that the minimal cost network has a branching structure since the norm favors higher multiplicity edges, representing shared transport. In this thesis, we construct a continuous flow that evolves some initial such network to reduce transport cost without altering its supply and demand distributions. Instead of limiting our scope to transport networks, we construct this $\mathbf{M}^\alpha$ mass reducing flow for real-valued flat chains by finding a real current of locally finite mass with the property that its restrictions are flat chains; the slices of such a restriction dictate the flow. Keeping the boundary fixed, this flow reduces the $\mathbf{M}^\alpha$ mass of the initial chain and is Lipschitz continuous under the flat-α norm. To complete the thesis, we apply this flow to transportation networks, showing that the flow indeed evolves branching transport networks to be more cost efficient.

Acknowledgments

> It was the best of times, it was the worst of times, it was the age of wisdom, it was the age of foolishness, it was the epoch of belief, it was the epoch of incredulity, it was the season of Light, it was the season of Darkness, it was the spring of hope, it was the winter of despair.
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> *Charles Dickens*
> *A Tale of Two Cities*

First and foremost, I would like to thank Bob Hardt for his guidance and encouragement. As we teetered back and forth between the age of wisdom and the age of ~~foolishness~~ goofiness, his clever sense of humor made the many hours spent in his office the best of times during my season of graduate school.

Of course, I need to thank Rice University's mathematics department for their support. I thank my committee members, Mike Wolf and Beatrice Riviere, for their thoughts and suggestions on this thesis. I also thank Tony Varilly-Alvarado and Tim Cochran for their words of encouragement, providing me a spring of hope.

I would like to thank my friends and family for their unconditional love and unwavering faith in me. They provide daily blessings that keep me in the season of Light. Finally, I must thank Him "who is able to do *immeasurably* more than all we ask or imagine" (Ephesians 3:20).

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CHAPTER 1

Introduction

1. An overview of geometric flows

By evolving objects (often generalized manifolds) by a gradient flow of some functional, geometric flows help solve optimization problems within calculus of variations. For example, one may wish to minimize the area functional by evolving hypersurfaces via the gradient flow for surface area. This leads to the mean curvature flow and its one dimensional version, called the curve shortening flow [**Whi02, CMP15, Gra87, GH86, Hui84**].

Other mathematicians have extended the ideas of the mean curvature flow to study various mass reducing flows for more generalized manifolds. Brakke created a mean curvature flow for varifolds [**Bra78**]. Evans and Spruck evolved level sets of viscosity solutions of suitable differential equations [**ES91, ES92a, ES92b, ES95**]. Others have used a time discretization process to create minimizing flows. For example, Cheng developed a mass reducing flow for integral currents [**Che93**]. Almgren and Taylor created a flat flow defined by a variety of integrands including crystalline in [**AT95**] and later with Wang in [**ATW93**]. In 2004, Haga, Hoshino, and Kikuchi constructed a harmonic map flow also following a time discretization process [**HHK04**].

Whether locally-defined flows like the mean curvature flow or limit-defined flows like Cheng's mass reducing flow, the underlying theme of geometric flows is to gain insight into difficult optimization problems. Instead of attempting to solve the optimization problem outright, one may be able to flow some initial object to improve its functional value. For the flow constructed in this thesis, we are motivated by evolving a transportation network in such a way to decrease cost.

2. Motivation for this flow: Optimal transportation theory

Transport theory as in the Monge-Kantorovich problem (1781; 1942) focuses on minimizing transportation cost in a way that depends only on optimally allocating or distributing goods [**Mon81, Kan42**]. Here, the cost of a transport plan is typically an integral of some convex function of distance, such as $|x-y|^p$ for $1 \leq p < \infty$. And so, the functional does not account for the actual path traveled by the goods; one can assume each good is transported along a straight line connecting its source to its destination.

While this model allows for easier analysis, it is not realistic for many practical problems. As noted by Xia [**Xia03**], the cost of transportation in real applications is rarely completely determined by the distance between the starting and ending positions. Instead, the physical transport path influences total cost because “carpooling” encourages overlapping in cost efficient ways, making ramified structures preferable. In other words, “Y shaped” paths can be more cost effective than “V shaped” ones. This is due to the fact that in operations there are often efficiencies of scale—i.e., the cost per pound for transporting freight decreases as the amount of freight increases.

One finds examples of such ramified paths throughout nature (e.g., tree root systems and cardiovascular systems), in addition to economics. In this vein, the actual path taken should influence cost so as to encourage overlapping structures for cost efficiency in carpooling. However, finding the optimal branch structure proves difficult, especially when studying the transportation of complicated goods like fractals, which may be represented by generalized Radon measures. In fact, algorithmically finding optimal concave cost networks, even with a finite number of sources and sinks, is NP-hard [**BZ15**].

Inspired by finding optimal transport paths, we create a minimizing flow that fixes the boundary and encourages overlapping structures by using an $\mathbf{M}^\alpha$ norm. Instead of restricting our attention to transportation networks, we conveniently develop a flow for real-valued, k-dimensional flat chains, which are domains of integration generalizing oriented manifolds, as conceptualized by Hassler Whitney in 1957 [**Whi57**]. Moreover, the resulting flow is both $\mathbf{M}^\alpha$ mass reducing and Lipschitz continuous in time with respect to a flat-α norm. The main theorem can be summarized as follows:

Main Theorem. Given a real-valued flat chain T_0 with finite $\mathbf{M}^\alpha$ mass, finite $\mathbf{M}^\alpha$ boundary mass, and compact support contained in some compact, star-shaped set $K \subset \mathbb{R}^m$, there exists a flow giving real-valued, k-dimensional flat chains T_t for almost every $t \geq 0$ so that $\partial T_t = \partial T_0$ and for almost every $r > t \geq 0$,

$$\mathbf{M}^\alpha(T_r) \leq \mathbf{M}^\alpha(T_t) \leq \mathbf{M}^\alpha(T_0).$$

Additionally,

$$\mathbf{F}_K^\alpha(T_r - T_t) \leq (r - t)\mathbf{M}^\alpha(T_0)$$

for almost every $r > t \geq 0$, where $\mathbf{F}_K^\alpha$ is the flat-α norm on chains.

The idea of the proof is to generalize Cheng's work for integral currents [**Che93**]. Using the direct method in the calculus of variations, we first construct discrete sequences that minimize a functional related to a scaled flat-α norm. Then, via a Cantor diagonalization argument and Brian White's compactness theorem [**Whi99b**], we find a $(k+1)$-dimensional, real-valued current of locally finite mass with the property that its restrictions (in time) are flat chains; the slices of such restrictions dictate the flow.

After discussing the flow's relation to the mean curvature flow, we explore applications to transportation networks. Since by construction the flow encourages mass-saving ramified structures, it is readily applicable to minimizing transportation costs. We flow an initial transport path to a local minimum by viewing transportation networks as 1-dimensional, real-valued flat chains and their transport costs as their $\mathbf{M}^\alpha$ masses, as in Bernot, Caselles, and Morel's book [**BCM09**]and in the work of Xia [**Xia03, Xia15, Xia05, Xia11a, XV10, XX10, Xia04, Xia14, Xia07, Xia11b**]. In fact, this flow allows for changes in the topology of the network, meaning that the network can evolve to include cost-efficient overlapping. Figure 1.1 is provided for a motivating example. Concretely, we will see the path evolving from a V-shaped initial path (sending goods from two locations to one location) to the optimal Y-shaped path through a "zipping up" mechanism.

The thesis is broken into two parts. Part 1 focuses on the background information for flat chains and the construction of the flow. Then, we focus on applying the flow to transportation networks in Part 2.

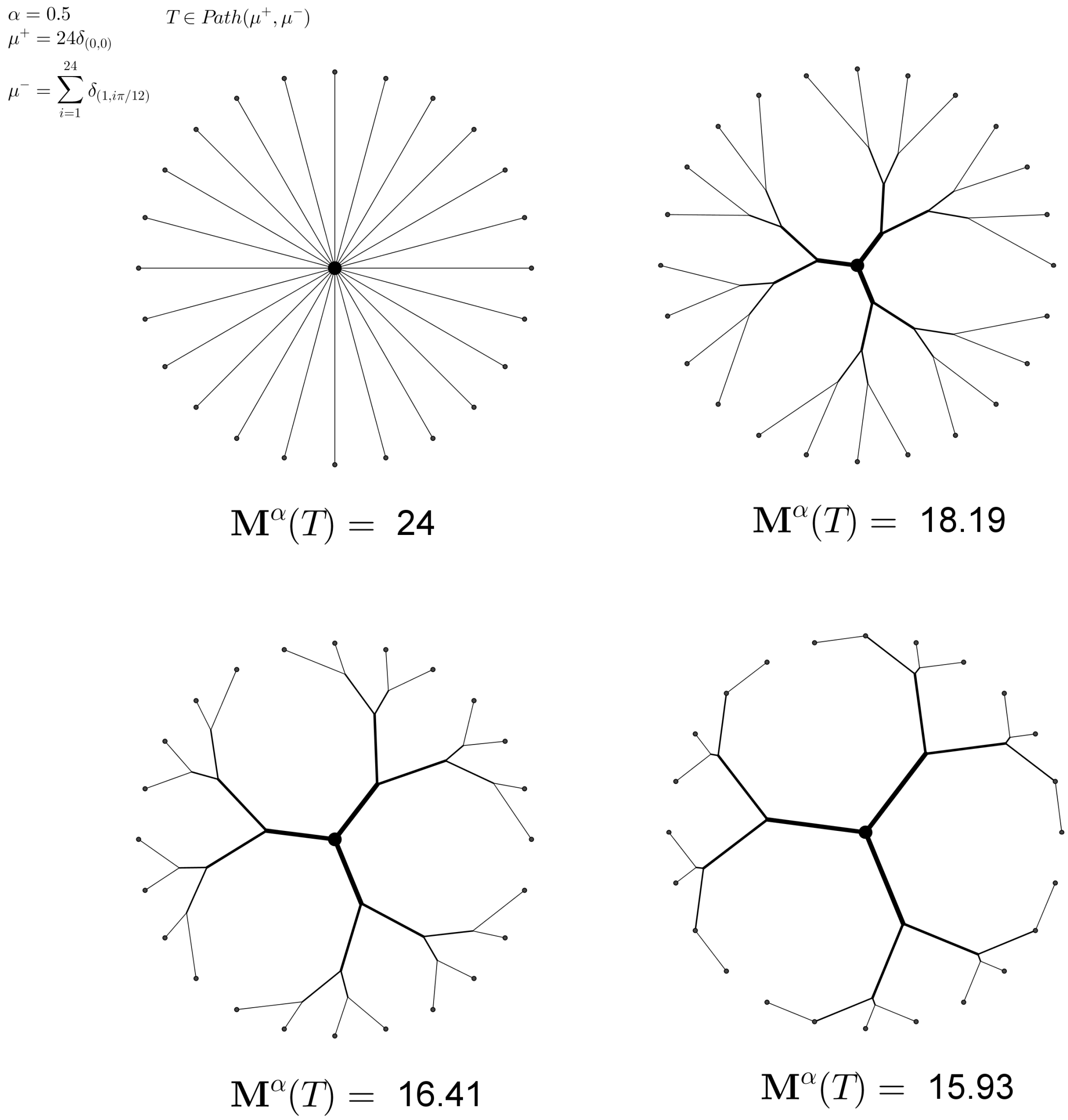


FIGURE 1.1. Here are snapshots of the flow on a transport path from one central Dirac mass (of weight twenty-four) to twenty-four equally spaced unit Dirac masses; here $\alpha = 0.5$.

Part 1

A Mass Minimizing Flow for Real-Valued Flat Chains

CHAPTER 2

Real-Valued Flat Chains in Euclidean Space

Flat chains generalize the concept of oriented manifolds, possibly with boundary. Their definition originates from investigating the following problem:

THE PLATEAU PROBLEM. Given a closed, $(k-1)$-dimensional boundary $\Gamma \subset \mathbb{R}^m$, find a k-dimensional surface S of least k-dimensional area among all surfaces with $\partial S = \Gamma$.

This problem was inspired by the Belgian physicist Joseph Plateau's (1801-1883) experiments with soap films [**Pla73**]. In fact, questions about existence and regularity for a k-dimensional Plateau solution became the key motivation for the modern development of the field of geometric measure theory.

To show the existence of a minimal surface S with the desired boundary, one might initially try to implement the direct method of calculus of variations—taking a minimizing sequence of surfaces with boundary Γ and extracting a subsequence converging to the solution surface. This would require existence of finite area surfaces with boundary Γ, lower semicontinuity of area with respect to some topology, and compactness with respect to this topology. (Additionally, one needs continuity of the boundary operator to ensure that the boundary stays fixed.) However, Figure 2.1 demonstrates how compactness of surfaces in $\mathbb{R}^3$ can go awry.

Hence, mathematicians had to develop new strategies to answer the Plateau problem. In the 1930's, Douglas and Rado independently solved the problem for

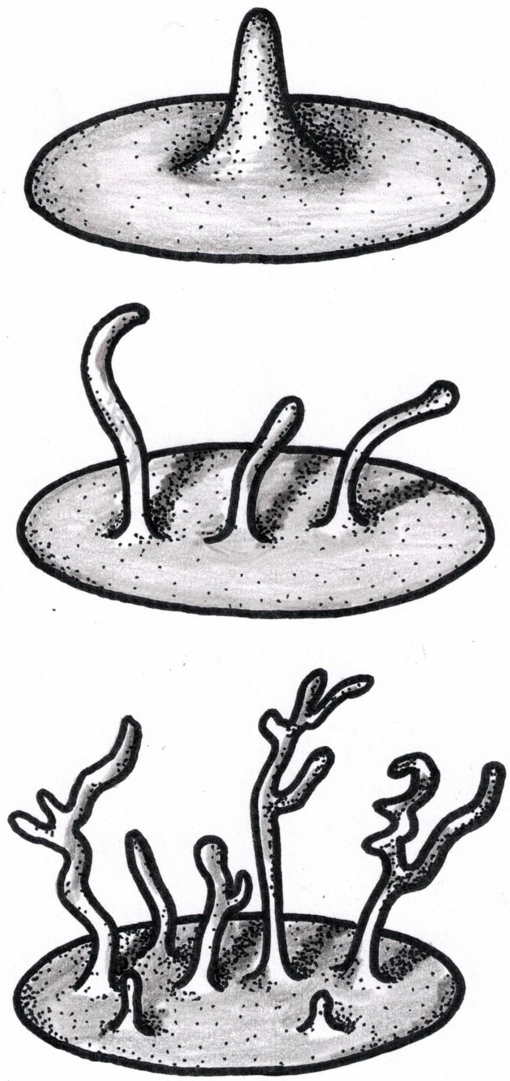


Figure 2.1. The space of surfaces (e.g. mappings) may fail to be compact with respect to the considered topology. Depicted above is a sequence of surfaces with the unit circle as boundary whose surface areas decrease towards π. However, since each surface gains more and more thin tentacles, the limit could include all of $\mathbb{R}^3$. See [**Mor09**].

2-dimensional surfaces that can be parametrized by a single disk; in particular, they showed that every smooth Jordan curve bounds a mapped-in disc of least mapping area [**Dou39, Rad30**]. Since their work heavily relies on conformal parameterizations of surfaces, it limits the types of possible singularities in solutions. In 1960, Reifenberg solved a formulation of the Plateau problem, but subject to certain topological constraints [**Rei60**]. A natural objection to these solutions is that one must eliminate conceivable solutions that form in actual soap film experiments because of a priori imposed assumptions on achievable singularities and topological complexities.

To circumvent many of these restrictions, Federer and Fleming offered a solution to the Plateau problem for arbitrary dimension and codimension by using their newly defined normal and integral currents [**FF60**]. Fleming gave another approach in 1966 using different generalized oriented manifolds, called flat chains [**Fle66**]. While avoiding the a priori assumptions on singularities and topological complexities, both currents and flat chains offer natural topologies with compactness properties, and thus, surmount the issue in implementing the direct method.

For more information on the history of the Plateau problem and its contribution to the founding of geometric measure theory, see [**AJ66, Mor09**].

One may consider finding an optimal transportation network as solving an $\mathbf{M}^\alpha$ mass generalization of the 1-dimensional Plateau problem. Motivated by this, we create an $\mathbf{M}^\alpha$ mass reducing flow on real-valued flat chains. In this chapter, we define flat chains and develop the preliminary material necessary for creating the flow. Sources that provide much of the background presented here include [**Fed69, Fle66, Mor09, Whi57, Whi99b, Whi99a, DPH12**].

1. Flat chains, $\mathbf{M}^\alpha$ mass, and the flat-α seminorm

Let $K \subset \mathbb{R}^m$ be a compact set in m-dimensional Euclidean space and let $k \in \{0, 1, \ldots, m\}$.

Definition 2.1. Let $\mathcal{P}_k(K)$ denote the group of polyhedral chains of dimension k with real coefficients and support in K. We may write any polyhedral chain in $\mathcal{P}_k(K)$ as $P = \sum g_i\sigma_i$ where each $g_i \in \mathbb{R}$ and $\{\sigma_i\}$ is a set of non-overlapping, oriented, k-dimensional simplices in K. For any $\alpha \in [0, 1]$, define the $\mathbf{M}^\alpha$ mass of P to be

$$\mathbf{M}^\alpha(P) := \sum |g_i|^\alpha \mathcal{H}^k(\sigma_i),$$

where $\mathcal{H}^k$ is the k-dimensional Hausdorff measure.

Define the *flat-α norm* of a polyhedral chain $P \in \mathcal{P}_k(K)$ by

$$\mathbf{F}^\alpha_K(P) := \inf\{\mathbf{M}^\alpha(P - \partial R) + \mathbf{M}^\alpha(R) : R \in \mathcal{P}_{k+1}(K)\},$$

where ∂ is the usual boundary operation on polyhedral chains. The flat-α distance between $P_1, P_2 \in \mathcal{P}_k(K)$ is $\mathbf{F}^\alpha_K(P_1 - P_2)$. Note that, by definition, $\mathbf{F}^\alpha_K(\partial P) \leq \mathbf{F}^\alpha_K(P)$.

At first glance, it may seem more natural to consider the topology induced by the $\mathbf{M}^\alpha$ mass instead of creating the flat-α norm. However, the following example elucidates why the flat-α norm is preferable: it more accurately captures the concept of geometric closeness for chains.

Example 2.2. Consider polyhedral chains $\{P_i\}$ and P each consisting of a directed edge of length 1 and weight 1. Suppose that each P_i is parallel to P and at

(Euclidean) distance 2^{-i} from P. Since $\mathbf{M}^\alpha(P_i - P_\epsilon) = 2$ for every $i \geq 0$, the sequence $\{P_i\}$ fails to converge to P in the topology induced by the $\mathbf{M}^\alpha$ norm. However, the sequence does converge to P in the flat-α norm since

$$\mathbf{F}_K^\alpha(P_i - P) \leq \mathbf{M}^\alpha(P_i - P - \partial Q) + \mathbf{M}^\alpha(Q) = (2)^{-i-1} + 2^{-i},$$

where Q is the two-dimensional chain as in Figure 2.2.

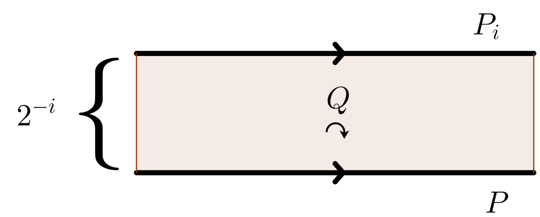


Figure 2.2. Topology induced by the $\mathbf{M}^\alpha$ norm vs. the flat-α norm

An additional benefit of the flat-α norm is the lower semicontinuity of the $\mathbf{M}^\alpha$ mass:

Lemma 2.3. [**Fle66, Whi99b**] *On $\mathcal{P}_k(K)$, the $\mathbf{M}^\alpha$ mass is lower semicontinuous with respect to the flat-α norm.*

Remark. Note that here we are thinking of the chain coefficients as elements of the group $G = \mathbb{R}$ with group norm $|g| = |g|^\alpha$ as in [**Fle66, Whi99b**].

Using the flat-α completion of $\mathcal{P}_k(K)$, we may now define general flat chains and their boundaries.

Definition 2.4. Denote by $\mathcal{F}_k(K)$ the $\mathbf{F}_K^\alpha$-completion of $\mathcal{P}_k(K)$ and call its elements *real-valued, k-dimensional flat chains.* Consider a sequence $\{P_i\} \subset \mathcal{P}_k(K)$ converging to T in the flat-α norm and note that

$$\mathbf{F}_K^\alpha(\partial P_i - \partial P_j) = \mathbf{F}_K^\alpha(\partial(P_i - P_j)) \leq \mathbf{F}_K^\alpha(P_i - P_j).$$

Thus, $\{\partial P_i\}$ is a convergent sequence of $(k-1)$-dimensional polyhedral chains and we say that its limit is a $(k-1)$-dimensional flat chain called the *boundary* of T, denoted ∂T.

In order to preserve the lower semicontinuity property, we define the $\mathbf{M}^\alpha$ mass of a real-valued flat chain as follows.

Definition 2.5. For any real flat chain $T \in \mathcal{F}_k(K)$, define its $\mathbf{M}^\alpha$ *mass* to be

$$\mathbf{M}^\alpha(T) \coloneqq \inf\{\liminf_{i\to\infty} \mathbf{M}^\alpha(P_i) : \{P_i\} \xrightarrow{\mathbf{F}_K^\alpha} T\}.$$

This definition indeed coincides with the one above when T is polyhedral. And in general, we may define the *flat-α norm* of T as

$$\mathbf{F}_K^\alpha(T) \coloneqq \inf\{\mathbf{M}^\alpha(T - \partial R) + \mathbf{M}^\alpha(R) : R \in \mathcal{F}_{k+1}(K)\}.$$

Lemma 2.6. *On $\mathcal{F}_k(K)$, the $\mathbf{M}^\alpha$ mass is lower semicontinuous with respect to the flat-α norm.*

□

Comparing the flat-α norm and the norm induced by the $\mathbf{M}^\alpha$ mass on real-valued flat chains, one finds that convergence in the $\mathbf{M}^\alpha$ norm is stronger than that in the flat-α norm when $k < n$ since $\mathbf{F}_K^\alpha(T) \le \mathbf{M}^\alpha(T)$. Example 2.2 provides an instance where convergence in the flat-α norm does not imply convergence in the $\mathbf{M}^\alpha$ mass norm.

Definition 2.7. For convenience, we define $\mathbf{N}^\alpha(T) := \mathbf{M}^\alpha(T) + \mathbf{M}^\alpha(\partial T)$ for $T \in \mathcal{F}_k(K)$; and let $\mathcal{N}_k^\alpha(K) := \{T \in \mathcal{F}_k(K) : \mathbf{N}^\alpha(T) < \infty\}$.

2. Compactness

Recall that Fleming defined flat chains and the flat-α norm in order to have the necessary compactness property for the direct method of calculus of variations. Using a deformation theorem shown in [**Whi99a**], White proved a more generalized compactness theorem for chains in [**Whi99b**]:

Theorem 2.8. *The set of flat chains of finite generalized mass and boundary mass with coefficients in an abelian group G is compact with respect to the generalized flat norm if and only if G has the property that balls are compact in G, i.e. $\{g \in G : |g| \le R\}$ is compact for all $R > 0$.*

Rephrasing this theorem into the context of real-valued flat chains, we have

Theorem 2.9. *Let $\alpha \in (0,1]$. Suppose $\{T_j\} \subset \mathcal{F}_k(K)$ with $\sup_{j\ge1} \mathbf{N}^\alpha(T_j) < \infty$. Then there exists a subsequence of $\{T_j\}$ converging in the flat-α distance to a real flat chain $T \in \mathcal{F}_k(K)$.*

Remark. Compactness fails in the case $\alpha = 0$ since $\{g \in \mathbb{R} : |g|^0 \leq R\} = \mathbb{R}$ for all $R > 0$.

In the following chapter, we employ the direct method of calculus of variations at several points in the construction of the flow, and so, we will often refer to Theorem 2.9. Consequently, the $\mathbf{M}^\alpha$ reducing flow will be restricted to $\alpha \in (0, 1]$.

3. Cartesian product of flat chains

While we cannot give a satisfactory definition for the Cartesian product of two arbitrary flat chains, we may provide one for when at least one of the chains is in $\mathcal{N}_k^\alpha(K)$. To do so, we must first define the Cartesian product of two polyhedral chains and then give a bound on the flat-α norm of their product.

Definition 2.10. Suppose P is a k-dimensional polyhedral chain in $\mathbb{R}^m$ and Q is an ℓ-dimensional polyhedral chain in $\mathbb{R}^n$. Their *Cartesian product*, denoted $P \times Q$, is a $(k+\ell)$-dimensional polyhedral chain in $\mathbb{R}^{m+n}$ and $\mathbf{M}^\alpha(P \times Q) = \mathbf{M}^\alpha(P)\mathbf{M}^\alpha(Q)$ since $|g_i g_j|^\alpha = |g_i|^\alpha |g_j|^\alpha$ for $\alpha \in [0, 1]$. Moreover, $\partial(P \times Q) = (\partial P) \times Q + (-1)^k P \times \partial Q$.

Lemma 2.11. [**Fle66**] *Let $\alpha \in [0, 1]$. For P and Q as above, $\mathbf{F}_K^\alpha(P \times Q) \leq \mathbf{N}^\alpha(P)\mathbf{F}_K^\alpha(Q)$ and $\mathbf{F}_K^\alpha(P \times Q) \leq \mathbf{F}_K^\alpha(P)\mathbf{N}^\alpha(Q)$*

Proof. To prove this, write $Q = R + \partial S$ for some ℓ-dimensional polyhedral chain R and some $(\ell+1)$-dimensional polyhedral chain S. Since $P \times \partial S = \pm[\partial P \times S - \partial(P \times S)]$, we have

$$\mathbf{F}_K^\alpha(P \times \partial S) \leq \mathbf{M}^\alpha(P \times S) + \mathbf{M}^\alpha(\partial P \times S),$$

$$\leq \mathbf{N}^\alpha(P)\mathbf{M}^\alpha(S).$$

Thus,

$$\begin{aligned}\mathbf{F}_K^\alpha(P \times Q) &\le \mathbf{F}_K^\alpha(P \times R) + \mathbf{F}_K^\alpha(P \times \partial S),\\ &\le \mathbf{M}^\alpha(P)\mathbf{M}^\alpha(R) + \mathbf{N}^\alpha(P)\mathbf{M}^\alpha(S),\\ &\le \mathbf{N}^\alpha(P)(\mathbf{M}^\alpha(R) + \mathbf{M}^\alpha(S)).\end{aligned}$$

The first desired inequality follows from the fact that R, S are arbitrary such polyhedral chains. The second inequality may be proved in the same manner as the first. □

Definition 2.12. Now we may define the *Cartesian product* of $T \in \mathcal{N}_k^\alpha(\mathbb{R}^m)$ and $S \in \mathcal{F}_\ell(\mathbb{R}^n)$. First, let $\{P_j\} \subset \mathcal{P}_k(\mathbb{R}^m)$ converge to T in the flat-α norm; then by the above lemma, we have for any $Q \in \mathcal{P}_\ell(\mathbb{R}^n)$

$$\mathbf{F}_K^\alpha[(P_i \times Q) - (P_j \times Q)] = \mathbf{F}_K^\alpha[(P_i - P_j) \times Q] \le \mathbf{F}_K^\alpha(P_i - P_j)\mathbf{N}^\alpha(Q).$$

So, $\{P_j \times Q\} \subset \mathcal{P}_{k+\ell}(\mathbb{R}^{m+n})$ converges and its limit is $T \times Q$. Now take $\{Q_i\} \subset \mathcal{P}_\ell(\mathbb{R}^n)$ converging to S in the flat-α norm. Since we may assume that $\mathbf{N}^\alpha(P_j) \to \mathbf{N}^\alpha(T)$ (see [**Fle66**]), we can prove Lemma 2.11 for T in place of P. Then because

$$\mathbf{F}_K^\alpha[T \times (Q_i - Q_j)] \le \mathbf{N}^\alpha(T)\mathbf{F}_K^\alpha(Q_i - Q_j),$$

we again have a convergent sequence $\{T \times Q_i\}$ and its limit is $T \times S$. Note that, for this sequence to converge, we indeed need $\mathbf{N}^\alpha(T) < \infty$.

We know the operation $\times$ is bilinear and $\partial(T \times S) = \partial T \times S + (-1)^k T \times \partial S$. Also, $\mathbf{M}^\alpha(T \times S) \le \mathbf{M}^\alpha(T)\mathbf{M}^\alpha(S)$.

For the construction of the flow, we are particularly interested in the case when $T = [|r,t|] \in \mathcal{F}_1(\mathbb{R})$ for $r < t$. Here,

$$\partial([|r,t|] \times S) = [|t|] \times S - [|r|] \times S - [|r,t|] \times \partial S.$$

Additionally,

$$\mathbf{M}^\alpha([|r,t|] \times S) = (t-r)\mathbf{M}^\alpha(S);$$

and

$$\begin{aligned}\mathbf{M}^\alpha(\partial([|r,t|] \times S)) &= \mathbf{M}^\alpha([|t|] \times S - [|r|] \times S - [|r,t|] \times \partial S), \\ &= 2\mathbf{M}^\alpha(S) + (t-r)\mathbf{M}^\alpha(\partial S).\end{aligned}$$

4. Push forward of real flat chains by Lipschitz maps

Since the construction of the flow will require a "birthday cake" decomposition, we will need finite $\mathbf{M}^\alpha$ mass cycles to bound higher dimensional chains achieving the considered norm. For integral chains (i.e., chains with integer coefficients), this fact would hold for the flat-α norm because of the isoperimetric inequality [**Che93**]. However, real-valued chains do not possess an isoperimetric property. So, we will define a new norm, called the fill-α norm, using cones over chains. Then, we will show that finite cycles bound higher dimensional chains achieving this new norm. In order to define cones over chains, we will need to first define push forwards of real-valued chains and give the homotopy formula.

If $g : \mathbb{R}^m \to \mathbb{R}^n$ is a piecewise linear map, then it induces a chain homomorphism $g_\sharp : \mathcal{P}_k(\mathbb{R}^m) \to \mathcal{P}_k(\mathbb{R}^n)$. This map may be extended to $g_\sharp : \mathcal{F}_k(\mathbb{R}^m) \to \mathcal{F}_k(\mathbb{R}^n)$.

Similarly, given a Lipschitz map $f : \mathbb{R}^m \to \mathbb{R}^n$, we may extend it to a homomorphism $f_\sharp : \mathcal{F}_k(\mathbb{R}^m) \to \mathcal{F}_k(\mathbb{R}^n)$ by approximating it by piecewise linear maps.

LEMMA 2.13. [**Fle66**] *Let $\alpha \in [0,1]$. If $f : \mathbb{R}^m \to \mathbb{R}^n$ is a proper Lipschitz map, then*

$$\mathbf{M}^\alpha(f_\sharp T) \leq (\mathbf{Lip} f)^n \mathbf{M}^\alpha(T),$$

and

$$\mathbf{F}_K^\alpha(f_\sharp T) \leq \sup\{(\mathbf{Lip} f)^n, (\mathbf{Lip} f)^{m+1}\}\mathbf{F}_K^\alpha(T),$$

for all $T \in \mathcal{F}_k(\mathbb{R}^m)$.

5. The homotopy formula

Let $K \subset \mathbb{R}^m$ be compact and let $f, g : K \subset \mathbb{R}^m \to \mathbb{R}^m$ be smooth mappings. The affine homotopy of f to g is the map $h : [0,1] \times K \to \mathbb{R}^m$, where $h(0,x) = f(x)$ and $h(1,x) = g(x)$ for all $x \in K$. If $T \in \mathcal{F}_k(K)$ has compact support and if the restriction of h to $[0,1] \times \mathrm{spt} T$ is proper, then $h_\sharp([|(0,1)|] \times T)$ is well-defined and

$$\partial h_\sharp([|(0,1)|] \times T) = g_\sharp T - f_\sharp T - h_\sharp([|(0,1)|] \times \partial T).$$

Rearranging the above equation gives the *homotopy formula*:

$$g_\sharp T - f_\sharp T = \partial h_\sharp([|(0,1)|] \times T) + h_\sharp([|(0,1)|] \times \partial T). \tag{1}$$

6. Cones over flat chains

Fix $x_0 \in K$ and without loss of generality, assume K is star-shaped with respect to the point x_0, (i.e., $tx + (1-t)x_0 \in K$ for each $x \in K$ and each $0 \le t \le 1$).

DEFINITION 2.14. Let $T \in \mathcal{F}_k(K)$ with compact support. Define $h : \mathbb{R} \times K \to \mathbb{R}^m$ by $h(t,x) = tx + (1-t)x_0$. Then, the *cone over* T, denoted $\text{cone}(T)$, is given by $h_\sharp([|(0,1)|] \times T)$. It follows that $\text{cone}(T) \in \mathcal{F}_{k+1}(K)$, and by the homotopy formula,

$$\partial(\text{cone}(T)) = T - \text{cone}(\partial T). \tag{2}$$

LEMMA 2.15. *Let $\alpha \in [0,1]$. If $T \in \mathcal{F}_k(K)$ has compact support, then*

$$\mathbf{M}^\alpha(cone(T)) \le diam(K)\mathbf{M}^\alpha(T).$$

PROOF. This follows from

$$\mathbf{M}^\alpha(\text{cone}(T)) \le \mathbf{M}^\alpha([0, diam(K)] \times T) = diam(K)\mathbf{M}^\alpha(T).$$

$\square$

COROLLARY 2.16. *Every cycle of finite $\mathbf{M}^\alpha$ mass bounds a chain of finite $\mathbf{M}^\alpha$ mass. More precisely, if $T \in \mathcal{N}_k^\alpha(K)$ and $\partial T = 0$, then there exists an $S \in \mathcal{F}_{k+1}(K)$ with $\partial S = T$ and $\mathbf{M}^\alpha(S) < \infty$; namely, $S = cone(T)$.*

$\square$

7. Defining a filling norm for cycles

Now, we may define the fill-α norm, which will be used in the construction of the flow. As before, assume K is star-shaped with respect to the fixed point $x_0 \in K$.

Definition 2.17. For cycles $T \in \mathcal{F}_k(K)$ with compact support, define a *filling-α mass* for T as

$$\mathbf{Fill}_K^\alpha(T) \coloneqq \inf\{\mathbf{M}^\alpha(Q) : \partial Q = T \text{ with } \mathrm{spt} Q \in K\}.$$

For convenience, denote the cycles in $\mathcal{F}_k(K)$ by $\mathcal{Z}_k$; i.e.,

$$\mathcal{Z}_k(K) \coloneqq \{T \in \mathcal{F}_k(K) : \partial T = 0\}.$$

Lemma 2.18. *On $\mathcal{Z}_k(K)$, the $\mathbf{Fill}_K^\alpha$ mass is lower semicontinuous with respect to the flat-α norm.*

Proof. This follows directly from the lower semicontinuity of the $\mathbf{M}^\alpha$ mass in Lemma 2.6. □

Lemma 2.19. *Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a proper Lipschitz map, then*

$$\mathbf{Fill}_K^\alpha(f_\sharp T) \leq (\mathbf{Lip} f)^n \mathbf{Fill}_K^\alpha(T)$$

for all $T \in \mathcal{Z}_k(K)$.

Proof. This follows from Lemma 2.13. □

The following lemma shows that the fill-α norm and the flat-α norm are equivalent for cycles. Hence, either may be used for calculating the geometric "closeness" of two chains with the same boundary.

LEMMA 2.20. *For $T \in \mathcal{Z}_k(K)$ with finite $\mathbf{M}^\alpha$ mass, then*

$$\mathbf{F}^\alpha_K(T) \leq \mathbf{Fill}^\alpha_K(T) \leq c\mathbf{F}^\alpha_K(T),$$

where c is a constant depending only on $K \subset \mathbb{R}^m$.

PROOF. The first inequality follows from the fact that the infimum on the right hand side is taken over a subset of $(k+1)$-dimensional flat chains. For the second inequality, write $T = Q + \partial R$. Note that $\partial Q = 0$, and so, $Q = \partial(\text{cone}(Q))$ by Equation (2). Thus, $T = \partial(\text{cone}(Q) + R)$. By the triangle inequality and Lemma 2.15,

$$\mathbf{Fill}^\alpha_K(T) \leq \mathbf{M}^\alpha(\text{cone}(Q)) + \mathbf{M}^\alpha(R) \leq (diam(K))\mathbf{M}^\alpha(Q) + \mathbf{M}^\alpha(R).$$

Taking the infimum over all such Q, R proves the second inequality for constant $c := \max\{diam(K), 1\}$. □

When defining the discrete sequences in the flow, we will use the fill-α norm to capture the idea of "closeness" of two chains with the same boundary. Moreover, for the "birthday cake" decomposition, we will need the existence of a higher dimensional chain achieving the fill-α norm between the two sequential chains forming its boundary.

Theorem 2.21. *Let $\alpha \in (0,1]$. For every $T \in \mathcal{Z}_k(K)$ with finite $\mathbf{M}^\alpha$ mass, there exists an $S \in \mathcal{F}_{k+1}(K)$ with $\partial S = T$ and $\mathbf{M}^\alpha(S) = \mathbf{Fill}_K^\alpha(T)$.*

Proof. From Corollary 2.16, we know $\mathbf{Fill}_K^\alpha(T) = \inf\{\mathbf{M}^\alpha(Q) : \partial Q = T \text{ with } \mathrm{spt} Q \in K\} < \infty$, and so, we may take a minimizing sequence $\{S_i\} \subset \mathcal{F}_{k+1}(K)$ where $\partial S_i = T$ and $\mathbf{M}^\alpha(S_i) < \infty$ for each i. Since $\lim_{i\to\infty} \mathbf{M}^\alpha(S_i) = \mathbf{Fill}_K^\alpha(T) < \infty$, the supremum of $\{\mathbf{M}^\alpha(S_i)\}$ must be finite. Thus,

$$\sup_i\{\mathbf{N}^\alpha(S_i)\} = \sup_i\{\mathbf{M}^\alpha(S_i) + \mathbf{M}^\alpha(T)\} < \infty.$$

By the Compactness Theorem 2.9, there exists a subsequence of $\{S_i\}$ converging in the flat-α norm to some $S \in \mathcal{F}_{k+1}(K)$ with $\partial S = \lim_{i'\to\infty} \partial S_{i'} = T$. Additionally,

$$\mathbf{Fill}_K^\alpha(T) \le \mathbf{M}^\alpha(S) \le \lim_{i'\to\infty} \mathbf{M}^\alpha(S_{i'}) = \mathbf{Fill}_K^\alpha(T),$$

by the lower semicontinuity of $\mathbf{M}^\alpha$ mass with respect to the flat-α norm (Lemma 2.6). □

8. Slicing of real flat chains

The M^α mass reducing flow will be defined as a real current of locally finite mass with the property that its restrictions are flat chains; the slices of such restrictions dictate the flow. Hence, we must define and give some properties for slicing real-valued flat chains.

Definition 2.22. Let $T \in \mathcal{N}_k^\alpha(K)$, $t \in \mathbb{R}$, and $f : \mathbb{R}^m \to \mathbb{R}$ be Lipschitz. Then we may define the *slice of* T as

$$\langle T, f, t_+ \rangle \coloneqq -\partial(T \llcorner \{x : f(x) > t\}) + (\partial T) \llcorner \{x : f(x) > t\} \in \mathcal{F}_{k-1}(\mathbb{R}^m).$$

Lemma 2.23. [**Whi99b**] *Suppose that* $T \in \mathcal{N}_k^\alpha(K)$ *and* $p : \mathbb{R}^m \to \mathbb{R}$ *is an orthogonal projection. Then, for* $\alpha \in [0, 1]$,

$$\int_{\mathbb{R}} \mathbf{M}^\alpha(\langle T, p, t_+ \rangle) dt \leq \mathbf{M}^\alpha(T),$$

and

$$\int_{\mathbb{R}} \mathbf{F}_K^\alpha(\langle T, p, t_+ \rangle) dt \leq \mathbf{F}_K^\alpha(T).$$

Lemma 2.24. [**DPH12**] *Let* $T_j, T \in \mathcal{F}_k(K)$ *with* $\sup_j\{\mathbf{N}^\alpha(T_j)\} < \infty$ *and suppose that the sequence* $\{T_j\}$ *converges to* T *in the flat-*α *norm. Let* $p : \mathbb{R}^m \to \mathbb{R}$ *be an orthogonal projection. Then, for almost every* $t \in \mathbb{R}$, *there is a subsequence* j' *such that*

$$\sum_{j'=1}^{\infty} \mathbf{F}_K^\alpha(\langle T_{j'} - T, p, t_+ \rangle) < \infty;$$

hence,

$$\lim_{j' \to \infty} \mathbf{F}_K^\alpha(\langle T_{j'}, p, t_+ \rangle - \langle T, p, t_+ \rangle) = 0,$$

i.e., the slices of the $T_{j'}$ *converge in the flat-*α *norm to the slice of* T.

Proof. First, pass to a subsequence to ensure that $\sum_{j'=1}^{\infty} \mathbf{F}_K^{\alpha}(T_{j'} - T) < \infty$. The result then follows from the fact that

$$\int_{\mathbb{R}} \sum_{j'=1}^{\infty} \mathbf{F}_K^{\alpha}(\langle T_{j'} - T, p, t_+\rangle)dt = \sum_{j'=1}^{\infty} \int_{\mathbb{R}} \mathbf{F}_K^{\alpha}(\langle T_{j'} - T, p, t_+\rangle)dt$$

and Lemma 2.23. □

Corollary 2.25. *In the setting above, we have*

$$\mathbf{M}^{\alpha}(\langle T, p, t_+\rangle) \leq \liminf_{j' \to \infty} \mathbf{M}^{\alpha}(\langle T_{j'}, p, t_+\rangle)$$

for almost every $t \in \mathbb{R}$ due to lower semicontinuity (Lemma 2.6).

□

CHAPTER 3

An $\mathbf{M}^\alpha$ Mass Reducing Flow for Real-Valued Flat Chains

1. Main theorem

In this chapter, we construct an $\mathbf{M}^\alpha$ mass reducing flow for real-valued flat chains. The main theorem is repeated below:

Main Theorem. Given $T_0 \in \mathcal{N}_k^\alpha(K)$ a real-valued flat chain with compact support contained in some compact, star-shaped set $K \subset \mathbb{R}^m$, there exists a flow giving real-valued, k-dimensional flat chains T_t for almost every $t \geq 0$ so that $\partial T_t = \partial T_0$, and for almost every $r > t \geq 0$,

$$\mathbf{M}^\alpha(T_r) \leq \mathbf{M}^\alpha(T_t) \leq \mathbf{M}^\alpha(T_0).$$

Additionally,

$$\mathbf{F}_K^\alpha(T_r - T_t) \leq (r-t)\mathbf{M}^\alpha(T_0)$$

for almost every $r > t \geq 0$.

The above T_t will be slices of a higher dimensional current found by taking a limit of "birthday cake" decompositions created out of discrete sequences of real-valued flat chains. In each sequence, each chain minimizes a functional that tends to reduce $\mathbf{M}^\alpha$ mass while ensuring that each chain stays geometrically close to the previous one in the sequence.

2. Step minimizing sequence

For the rest of the chapter, we will always assume that $\alpha \in (0,1]$ and that $K \subset \mathbb{R}^m$ is compact and star-shaped with respect to some $x_0 \in K$.

Begin with a fixed step size $h > 0$ and let $T_0 \in \mathcal{N}_k^\alpha(K)$ have compact support. We will inductively define a sequence of real-valued flat chains $\{T_j^h\}$ so that $T_0^h = T_0$ and T_j^h (step with respect to h) minimizes the following functional:

$$G_j(T) = G_j(T_{j-1}^h, T, h) := [\mathbf{M}^\alpha(T)]^2 + \frac{[\mathbf{Fill}_K^\alpha(T_{j-1}^h - T))]^2}{h}, \tag{3}$$

among chains with $\partial T = \partial T_0$ and with compact support in K. The motivation for these functionals is to reduce the $\mathbf{M}^\alpha$ mass of a chain while keeping the new one "close" to the original, as ensured by the fill-α mass.

THEOREM 3.1. *For each positive integer j, there exists a T_j^h with support in K.*

PROOF. Consider the collection of real flat chains

$$\mathcal{C} = \{T : \partial T = \partial T_0, T \text{ has compact support in } K\},$$

which is non-empty since $T_0 \in \mathcal{C}$. Set $T_0^h = T_0$.

Now inductively define T_j^h. Assume that T_{j-1}^h is already defined and its support contained in K. Set

$$L_j := \inf\{G_j(T_{j-1}^h, T, h) : T \in \mathcal{C}\},$$

which is finite because $T_0 \in \mathcal{C}$. Also let $\{R_i\} \subset \mathcal{C}$ be a sequence such that

$$\lim_{i\to\infty} G_j(R_i) = L_j;$$

then, $\sup_i G_j(R_i) < \infty$.

Because of this and since $\{R_i\} \subset \mathcal{C}$, we have

$$\begin{aligned}\sup_i\{\mathbf{N}^\alpha(R_i)\} &= \sup_i\{\mathbf{M}^\alpha(R_i) + \mathbf{M}^\alpha(\partial R_i)\}, \\ &\leq \sup_i \left\{\sqrt{G_j(R_i)} + \mathbf{M}^\alpha(\partial T_0)\right\} < \infty.\end{aligned}$$

Thus, by the Compactness Theorem 2.9, there exists a subsequence $\{R_{i'}\}$ which converges in the flat-α distance to some real flat chain, say $\widehat{T_j^h}$. Since the $\mathbf{M}^\alpha$ and $\mathbf{Fill}_K^\alpha$ masses are lower semicontinuous with respect to the flat-α norm, we find

$$\begin{aligned}L_j \leq G_j(T_{j-1}^h, \widehat{T_j^h}, h) &= [\mathbf{M}^\alpha(\widehat{T_j^h})]^2 + \frac{[\mathbf{Fill}_K^\alpha(T_{j-1}^h - \widehat{T_j^h}))]^2}{h}, \\ &\leq \liminf_{i' \to \infty}[\mathbf{M}^\alpha(R_{i'})]^2 + \liminf_{i' \to \infty} \frac{[\mathbf{Fill}_K^\alpha(T_{j-1}^h - R_{i'}))]^2}{h}, \\ &\leq \lim_{i' \to \infty} G_j(R_{i'}) = L_j;\end{aligned}$$

and we also have

$$\partial \widehat{T_j^h} = \lim_{i' \to \infty} \partial R_{i'} = \partial T_0.$$

Now, using $\widehat{T_j^h}$, define T_j^h so that its support is contained in K. Let $f : \mathbb{R}^m \to K$ be the nearest point projection map so that

$$|f(x) - x| = \inf\{|y - x| : y \in K\}.$$

Note that f is a Lipschitz map with Lipschitz constant no greater than 1. Set $T_j^h := f_\sharp \widehat{T_j^h}$. By Lemma 2.13, Lemma 2.19, and the property that $f(x) = x$ for every $x \in K$, we know

$$
\begin{aligned}
G_j(T_{j-1}^h, T_j^h, h) &= [\mathbf{M}^\alpha(T_j^h)]^2 + \frac{[\mathbf{Fill}_K^\alpha(T_{j-1}^h - T_j^h))]^2}{h}, \\
&\le [\mathbf{M}^\alpha(\widehat{T_j^h})]^2 + \frac{[\mathbf{Fill}_K^\alpha(T_{j-1}^h - \widehat{T_j^h}))]^2}{h}, \\
&\le L_j.
\end{aligned}
$$

Moreover,

$$\partial T_j^h = \partial f_\sharp \widehat{T_j^h} = f_\sharp \partial \widehat{T_j^h} = f_\sharp \partial T_0 = \partial T_0.$$

□

Notice that we defined T_j^h in such a way that ensures

$$
\begin{aligned}
[\mathbf{M}^\alpha(T_j^h)]^2 &\le G_j(T_{j-1}^h, T_j^h, h), \\
&= [\mathbf{M}^\alpha(T_j^h)]^2 + \frac{[\mathbf{Fill}_K^\alpha(T_{j-1}^h - T_j^h))]^2}{h} \\
&\le G_j(T_{j-1}^h, T_{j-1}^h, h), \\
&= [\mathbf{M}^\alpha(T_{j-1}^h)]^2.
\end{aligned}
$$

Thus, we have the following results:

LEMMA 3.2. *The mass of the step minimizing sequence is non-increasing; i.e.*

$$\mathbf{M}^\alpha(T_j^h) \le \mathbf{M}^\alpha(T_{j-1}^h) \le \mathbf{M}^\alpha(T_0).$$

Additionally, we have the estimate

$$\mathbf{Fill}_K^\alpha(T_{j-1}^h - T_j^h) \le \sqrt{h([\mathbf{M}^\alpha(T_{j-1}^h)]^2 - [\mathbf{M}^\alpha(T_j^h)]^2)} \le \sqrt{h}\mathbf{M}^\alpha(T_0).$$

3. Construction of the flow

To construct the flow, we will find a locally finite mass current whose slices will dictate how to reduce the $\mathbf{M}^\alpha$ mass of the initial current.

For every step $h > 0$, we know from Lemma 3.2 that $\mathbf{Fill}^\alpha_K(T^h_{j-1} - T^h_j) < \infty$. Note that $\partial(T^h_{j-1} - T^h_j) = 0$ by construction. So, by Theorem 2.21, there exists a $(k+1)$-dimensional real-valued flat chain S^h_j satisfying $\partial S^h_j = T^h_{j-1} - T^h_j$ and

$$\mathbf{M}^\alpha(S^h_j) = \mathbf{Fill}^\alpha_K(T^h_{j-1} - T^h_j) \leq \sqrt{h}\mathbf{M}^\alpha(T_0). \tag{4}$$

Using a "birthday cake" decomposition, construct a $(k+1)$-dimensional real-valued current in $\mathbb{R}^+ \times \mathbb{R}^m$ as follows. Let

$$S_h := \sum_{j=1}^{\infty}(-[|(j-1)\sqrt{h}, j\sqrt{h}|] \times T^h_j + [|(j-1)\sqrt{h}|] \times S^h_j). \tag{5}$$

See Figure 3.1.

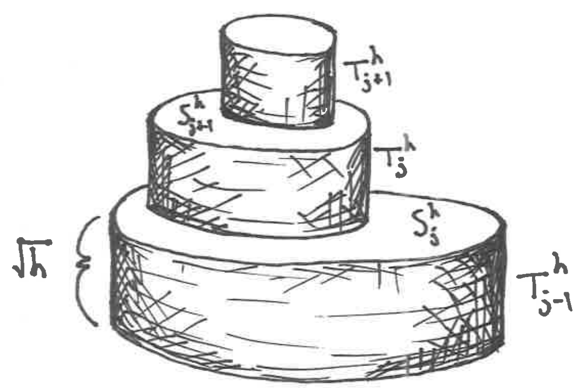


FIGURE 3.1. Shown here is a piece of the "birthday cake" decomposition of an $S_h \subset \mathbb{R}^+ \times \mathbb{R}^m$, for $h > 0$.

Computing the boundary of S_h with analogous techniques of Section 3, we find

$$\begin{aligned}
\partial S_h &= \sum_{j=1}^{\infty} \partial(-[|(j-1)\sqrt{h}, j\sqrt{h}|] \times T_j^h + [|(j-1)\sqrt{h}|] \times S_j^h), \\
&= \sum_{j=1}^{\infty}([|(j-1)\sqrt{h}|] \times T_j^h - [|j\sqrt{h}|] \times T_j^h \\
&\qquad + [|(j-1)\sqrt{h}, j\sqrt{h}|] \times \partial T_j^h + [|(j-1)\sqrt{h}|] \times \partial S_j^h), \\
&= \sum_{j=1}^{\infty}([|(j-1)\sqrt{h}|] \times T_j^h - [|j\sqrt{h}|] \times T_j^h \\
&\qquad + [|(j-1)\sqrt{h}, j\sqrt{h}|] \times \partial T_0 + [|(j-1)\sqrt{h}|] \times (T_{j-1}^h - T_j^h)), \\
&= \sum_{j=1}^{\infty}([|(j-1)\sqrt{h}|] \times T_{j-1}^h - [|j\sqrt{h}|] \times T_j^h) + \sum_{j=1}^{\infty}([|(j-1)\sqrt{h}, j\sqrt{h}|] \times \partial T_0), \\
&= [|0|] \times T_0 + [|0, +\infty|] \times \partial T_0.
\end{aligned}$$

Theorem 3.3. *Let S_i be an abbreviation for the real flat chain S_h with $h = 2^{-i}$. There exists a subsequence of $\{S_i\}$ converging to a locally finite real current S with $\partial S = [|0|] \times T_0 + [|0, +\infty|] \times \partial T_0$. Additionally, S has the property such that for every $r \geq 0$, $S^r := S \llcorner [0, r] \times \mathbb{R}^m$ is a finite mass flat chain.*

Proof. Let $\{r_n\}_{n=1}^{\infty} \subset \mathbb{R}^+$ be a sequence of monotonically increasing real numbers. Fix $r_n > 0$. In order to use the compactness theorem, we need to first restrict each S_i and ∂S_i to $[0, r_n] \times \mathbb{R}^m$. Note that these restricted objects are flat chains and call them $S_i^{r_n}$ and $\partial S_i^{r_n}$, respectively. We will show $\sup_i\{\mathbf{N}^{\alpha}(S_i^{r_n})\} < \infty$.

Fix $h = 2^{-i}$; choose integer l so that $(l-1)\sqrt{2^{-i}} < r_n \leq l\sqrt{2^{-i}}$. Using Lemma 3.2 and inequality 4, compute

$$\begin{aligned}
\mathbf{M}^\alpha(S_i^{r_n}) &\leq \sum_{j=1}^{l}(\mathbf{M}^\alpha(-[|(j-1)\sqrt{2^{-i}}, j\sqrt{2^{-i}}|] \times T_j^{2^{-i}}) + \mathbf{M}^\alpha([|(j-1)\sqrt{2^{-i}}|] \times S_j^{2^{-i}})),\\
&\leq \sum_{j=1}^{l}(\sqrt{2^{-i}}\mathbf{M}^\alpha(T_0) + \mathbf{M}^\alpha(S_j^{2^{-i}})),\\
&\leq \sum_{j=1}^{l}(\sqrt{2^{-i}}\mathbf{M}^\alpha(T_0) + \sqrt{2^{-i}}\mathbf{M}^\alpha(T_0)),\\
&\leq 2r_n\mathbf{M}^\alpha(T_0).
\end{aligned}$$

Next compute

$$\begin{aligned}
\mathbf{M}^\alpha(\partial S_i^{r_n}) &= \mathbf{M}^\alpha([|0|] \times T_0 + [|0, +\infty|] \times \partial T_0),\\
&\leq \mathbf{M}^\alpha(T_0) + \mathbf{M}^\alpha([|0, \lceil r_n \rceil|] \times \partial T_0),\\
&\leq \mathbf{M}^\alpha(T_0) + \lceil r_n \rceil \mathbf{M}^\alpha(\partial T_0).
\end{aligned}$$

Thus,

$$\sup_i\{\mathbf{N}^\alpha(S_i^{r_n})\} \leq (2r_n + 1)\mathbf{M}^\alpha(T_0) + \lceil r_n \rceil \mathbf{M}^\alpha(\partial T_0) < \infty.$$

By the Compactness Theorem 2.9, there exists a subsequence $\{S_{i'}^{r_n}\}$ converging to some real flat chain S^{r_n} in the flat-α norm.

Now consider the sequence $\{S_{i'}^{r_{n+1}}\}$. Iterating the above process, we again find a convergent subsequence $\{S_{i''}^{r_{n+1}}\}$ with limit $S^{r_{n+1}}$. Note that by construction $S^{r_n} = S^{r_{n+1}}$ in $[0, r_n] \times \mathbb{R}^m$. Continue this process.

Using a Cantor diagonalization argument, we find a well-defined, locally finite mass, real-valued current S defined on all of $\mathbb{R}^+ \times \mathbb{R}^m$. Additionally, S has the property that for each fixed time r, $S^r := S \llcorner ([0,r] \times \mathbb{R}^m) \in \mathcal{N}^\alpha_{k+1}(R^+ \times R^m)$. (This follows from the fact that the diagonal in the Cantor argument will eventually be a subsequence of a previous row.)

Note that the boundary condition follows from the fixed boundary for all S_i. $\square$

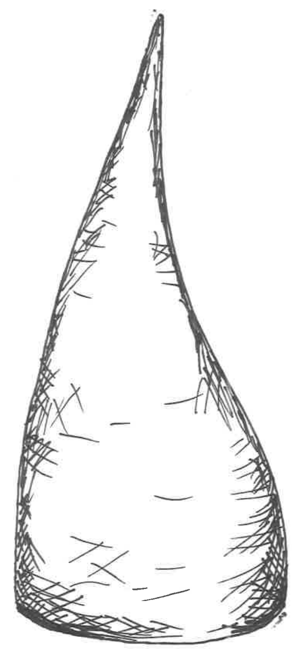


FIGURE 3.2. Shown here is a piece of the $(k+1)$-dimensional current $S \subset \mathbb{R}^+ \times \mathbb{R}^m$; the slices of suitable restrictions dictate the flow.

Let $p : \mathbb{R}^+ \times \mathbb{R}^m \to \mathbb{R}^+$ be the projection defined by $p(t,x) = t$. Define the slice of S by p at $t > 0$ by

$$\langle S, p, t_+ \rangle := \langle S^r, p, t_+ \rangle,$$

for fixed $r > t$. (This is well-defined since $\langle S^r, p, t_+ \rangle = \langle S^s, p, t_+ \rangle$ for every $s > t$ by the construction of S.) Note that this slice is a finite $\mathbf{M}^\alpha$ mass, k-dimensional flat chain for almost every $t > 0$ by the additional property of S. Similarly, we may define the slices of S_i for each i and again have $\langle S_i, p, t_+ \rangle \in \mathcal{N}^\alpha_{k+1}(\mathbb{R}^+ \times \mathbb{R}^m)$.

Set

$$T_t \coloneqq \pi_\sharp \langle S, p, t_+ \rangle$$

where $\pi : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ is the projection map defined by $\pi(t, x) = x$. We will utilize the Slicing Lemma 2.24 to prove the following theorems:

THEOREM 3.4. *For almost every $r > t \geq 0$,*

$$\mathbf{M}^\alpha(T_r) \leq \mathbf{M}^\alpha(T_t) \leq \mathbf{M}^\alpha(T_0)$$

and

$$\partial T_t = \partial T_0.$$

PROOF. Note that Lemma 3.2 already shows this result for when t is a dyadic rational number. Otherwise, by Equation (5),

$$\langle S_i, p, t_+ \rangle = [|t|] \times T_j^{2^{-i}},$$

where j is the largest integer such that $j \leq 2^{-i}t$. By the Slicing Lemma 2.24, we know that for almost every $t > 0$, there exists a subsequence i' such that

$$\mathbf{F}_K^\alpha(\langle S_{i'}, p, t_+ \rangle - \langle S, p, t_+ \rangle) \to 0.$$

Thus, for all such non-dyadic rational t, we have

$$\mathbf{F}_K^\alpha(\pi_\sharp(\langle S_{i'}, p, t_+\rangle) - \pi_\sharp(\langle S, p, t_+\rangle)) \to 0;$$

$$\mathbf{F}_K^\alpha(\pi_\sharp([|t|] \times T_j^{2^{-i'}}) - T_t) \to 0;$$

$$\mathbf{F}_K^\alpha(T_j^{2^{-i'}} - T_t) \to 0.$$

By the lower semicontinuity of $\mathbf{M}^\alpha$ mass with respect to the flat-α distance (Lemma 2.6) and by Lemma 3.2,

$$\mathbf{M}^\alpha(T_t) \leq \liminf_{i' \to \infty} \mathbf{M}^\alpha(T_j^{2^{-i}}) \leq \mathbf{M}^\alpha(T_0).$$

A similar argument shows that $\mathbf{M}^\alpha(T_r) \leq \mathbf{M}^\alpha(T_t)$.

Moreover, by construction,

$$\partial T_t = \lim_{i' \to \infty} \partial T_j^{2^{-i'}} = \partial T_0.$$

□

THEOREM 3.5. *The constructed flow is Lipschitz continuous in time with respect to the flat-α norm; i.e., for $0 \leq t < r$,*

$$\mathbf{F}_K^\alpha(T_r - T_t) \leq (r - t)\mathbf{M}^\alpha(T_0).$$

Proof. Let $0 \leq t < r$. Since $\partial S = [|0|] \times T_0 + [|0, +\infty|] \times \partial T_0$,

$$\langle S, p, r_+ \rangle - \langle S, p, t_+ \rangle = \partial\{S \llcorner \{(t,r) \times \mathbb{R}^m\}\} - (\partial S) \llcorner \{(t,r) \times \mathbb{R}^m\},$$

$$= \partial\{S \llcorner \{(t,r) \times \mathbb{R}^m\}\} - [|t,r|] \times \partial T_0.$$

Thus,

$$T_r - T_t = \pi_\sharp(\partial\{S \llcorner \{(t,r) \times \mathbb{R}^m\}\} - [|t,r|] \times \partial T_0),$$

$$= \partial \pi_\sharp(S \llcorner \{(t,r) \times \mathbb{R}^m\}).$$

Since the flat-α norm is lower semicontinuous, then for $\{i'\}$ as in the proof for Theorem 3.4, we have

$$\mathbf{F}^\alpha_K(T_r - T_t) \leq \mathbf{F}^\alpha_K\big(\partial \pi_\sharp(S \llcorner \{(t,r) \times \mathbb{R}^m\})\big),$$

$$\leq \liminf_{i' \to \infty} \mathbf{F}^\alpha_K\big(\partial \pi_\sharp(S_{i'} \llcorner \{(t,r) \times \mathbb{R}^m\})\big),$$

$$\leq \liminf_{i' \to \infty} \mathbf{M}^\alpha\big(\pi_\sharp(S_{i'} \llcorner \{(t,r) \times \mathbb{R}^m\})\big),$$

$$\leq (r-t)\mathbf{M}^\alpha(T_0).$$

To see the last inequality, choose l, l' so that

$$(l-1)\sqrt{2^{-i'}} < t \leq l\sqrt{2^{-i'}} \text{ and } (l'-1)\sqrt{2^{-i'}} < r \leq l'\sqrt{2^{-i'}};$$

then,

$$
\begin{aligned}
&\mathbf{M}^\alpha\big(\pi_\sharp(S_{i'} \llcorner \{(t,r) \times \mathbb{R}^m\})\big) \\
&\quad \le \mathbf{M}^\alpha\left(\pi_\sharp\left(\sum_{j=l}^{l'} -[|(j-1)\sqrt{2^{-i'}}, j\sqrt{2^{-i'}}|] \times T_j^{2^{-i'}} + [|(j-1)\sqrt{2^{-i'}}|] \times S_j^{2^{-i'}}\right)\right), \\
&\quad = \qquad \mathbf{M}^\alpha\Big(\sum_{j=l}^{l'} S_j^{2^{-i'}}\Big).
\end{aligned}
$$

The last equality follows from the definition of projection π and the fact that $\mathcal{H}^{k+1}(T_j^{2^{-i'}}) = 0$ (since each $T_j^{2^{-i'}}$ is k-dimensional). Finally, using the triangle inequality, Equation (4), and our choice of ℓ, ℓ', we find that the above is bounded by

$$
\sum_{j=l}^{l'} \sqrt{2^{-i'}}\mathbf{M}^\alpha(T_0) \le (r - t + \sqrt{2^{-i'}})\mathbf{M}^\alpha(T_0).
$$

$\square$

CHAPTER 4

Example: A Circle

Now, we will see how the flow acts on a circle in the plane. Since the flow reduces $\mathbf{M}^\alpha$ mass, we may foresee that a circle shrinks to a point under the flow. Surprisingly, the circle need not flow to its center, but may flow to any point within the circle, as discussed in the following chapter.

However, the real coefficients of the chains complicates this example since the chains could potentially "smear out" before extinguishing to a point. We will use rearrangements to show that one may assume that the constructed flow shrinks circles to a point in such a way that for almost every time $t > 0$, T_t is also a circle when $\alpha \in (0, 1/2]$. But first, we will give a general introduction to the required rearrangement: Schwarz symmetrization. We follow the presentation given in [**Kaw85**].

1. Rearrangements

1.1. Introduction. In general, rearrangement methods replace a given function u by a related function u^* having some desired property like symmetry or monotonicity. To construct u^*, one usually rearranges the level sets of u. Here, one may consider u^* as the hill resulting from u after some landscaping. There are many different choices of landscaping, each yielding a different rearrangement.

Rearrangements may be used to prove that solutions of some variational problem possess a particular desired property. Suppose that u is the unique minimizer of some functional J over a set of admissible functions. Set u^* to be the rearrangement of

u having this desired property and show that it is an admissible function. Then, proving that $J(u^*) \leq J(u)$ with equality if and only if $u = u^*$ shows that u equals u^*, and thus, it must have the desired property.

1.2. Schwarz symmetrization. The most common type of rearrangement is Schwarz symmetrization , also known as spherically symmetric rearrangement, named after H. A. Schwarz [**A.90**]:

DEFINITION 4.1. Let $D \subset \mathbb{R}^m$ be a compact set. We define the Schwarz symmetrization D^* of D by

$$D^* := \begin{cases} \text{a ball of the same } m\text{-dimensional Hausdorff measure as} & \text{if } D \neq \emptyset, \\ \quad D \text{ with center at the origin} & \\ \emptyset & \text{if } D = \emptyset. \end{cases}$$

REMARK. The *Schwarz symmetrization* D^* of a compact set D can also be obtained as a limit of repeated Steiner symmetrizations. Polya and Szegö [**PS51**] as well as Leichtweiss [**Lei80**] showed this for convex sets, while Brascamp, Lieb, and Luttinger proved this is true also for non-convex sets [**BLL74**].

While we only need to define rearrangements for sets, we give the definition for functions to complete this section.

DEFINITION 4.2. Let $u : \mathbb{R}^m \to \mathbb{R}^+$ be a Lipschitz continuous function with compact support $D \subset \mathbb{R}^m$. We rearrange u by rearranging its level sets $D_c := \{x \in$

$D : u(x) \geq c\}$ for $c \in \mathbb{R}$. Define the *Schwarz symmetrization* u^* of u as

$$u^* := \sup\{c \in \mathbb{R} : x \in D_c^*\} \text{ for } x \in D^*.$$

2. The flow on a circle

EXAMPLE 4.3. Let $\alpha \in (0, 1/2]$ and denote by $\mathbf{S}_{r_0}$ the circle of radius r_0 in $\mathbb{R}^2$ centered at the origin. Let $T_0 = [|\mathbf{S}_{r_0}|]$ be the flat chain oriented counter-clockwise with density 1. Then, each T_j^h may also be a circle with density 1, but of smaller radius, and hence, so is each T_t for almost every t. Additionally, the radius of T_t is given by $r(t) = \sqrt{r_0^2 - 2t}$.

PROOF. Inductively, we demonstrate that for each fixed step we may assume that the chains in each minimizing sequence are circles.

Fix $0 < h < 1/2$. Assume that T_{j-1}^h is a circle of radius $r_0 \in [0, 1]$ with density 1, and suppose T minimizes G_j. Set $S = T - T_{j-1}^h$; so, S minimizes the functional

$$\widetilde{G_j}(R) := [\mathbf{M}^\alpha(\partial R - T_{j-1}^h)]^2 + \frac{[\mathbf{M}^\alpha(R)]^2}{h},$$

where $R \in \mathcal{F}_2(\mathbf{D}_{r_0})$ has compact support and $\partial(\partial R) = 0$.

Now, let $\epsilon > 0$ and let $S_\epsilon \in \mathcal{P}_2(\mathbf{D}_{r_0})$ be an 2-dimensional polyhedral chain with compact support in $\mathbf{D}_{r_0}$ such that $\mathbf{M}^\alpha(S_\epsilon) < \mathbf{M}^\alpha(S) + \epsilon$. We may write $S_\epsilon = \sum_{i=1}^n g_i \sigma_i$ for $\{\sigma_i\}$ non-overlapping, oriented, 2-dimensional simplices in $\mathbf{D}_{r_0}$ and $g_i \in \mathbb{R}^+$ with $g_1 \leq g_2 \leq \cdots \leq g_n$.

We will use Schwarz rearrangement to find $S^*_\epsilon \in \mathcal{F}_2(\mathbf{D}_{r_0})$ consisting of concentric annuli. First, define

$$g(x) := \begin{cases} g_i & \text{if } x \in \sigma_i, \\ \\ 0 & \text{if otherwise.} \end{cases}$$

Set $\Omega_c := \{x \in K : g(x) \geq c\}$ for $c \in \mathbb{R}$. Rearrange the support of S_ϵ by rearranging each Ω_c as follows:

$$\Omega^*_c = \begin{cases} \text{ball of the same 2-dimensional Hausdorff measure as} & \text{if } \Omega_c \neq \emptyset, \\ \quad \Omega_c \text{ with center at the origin} & \\ \emptyset & \text{if } \Omega_c = \emptyset. \end{cases}$$

And now, define

$$S^*_\epsilon := \sum_{i=1}^{n} g_i \sigma^*_i.$$

By construction, $\{\sigma^*_i\}$ are concentric, oriented annuli and for each i, $\mathcal{H}^2(\sigma^*_i) = \mathcal{H}^2(\sigma_i)$. Thus,

$$\mathbf{M}^\alpha(S^*_\epsilon) = \sum_{i=1}^{n} |g_i|^\alpha \mathcal{H}^2(\sigma_i) = \mathbf{M}^\alpha(S_\epsilon) < \mathbf{M}^\alpha(S) + \epsilon.$$

By induction and by Lemma 4.4 (with appropriate scaling), we know that for annulus A with weight 1, outer radius equaling that of T^h_{j-1}, and inner radius equaling $\sqrt{r_0 - 2h}$,

$$\widetilde{G_j}(A) < \widetilde{G_j}(S) + c\epsilon.$$

Thus, $T^h_j = \partial S - T^h_{j-1}$ may be approximated by a circle with weight 1. Since this is for every flat chain in each discrete sequence, we may assume that each T_t for almost every $t > 0$ is a circle with weight 1.

Lemma 4.5 completes the example by showing that the radius of T_t is given by $r(t) = \sqrt{r_0^2 - 2t}$. $\square$

Lemma 4.4. *Let* $\alpha \in (0, 1/2]$ *and fix* $0 < h < 1/2$. *Suppose* S *is an annulus with weight* $d \in [0,1]$ *and inner radius* $r \in [0,1]$. *Let* $T_0 = [|\mathbf{S}_1|]$. *Define*

$$G(S) \coloneqq [\mathbf{M}^\alpha(\partial S - T_0)]^2 + \frac{[\mathbf{M}^\alpha(S)]^2}{h};$$

$$G((r,d)) \coloneqq [(1-d)^\alpha 2\pi + d^\alpha 2\pi r]^2 + \frac{[(1-r^2)d^\alpha \pi]^2}{h}.$$

See Figure 4.1. The minimum of G *is achieved by an annulus with inner radius* $r = \sqrt{1-2h}$ *and weight* $d = 1$.

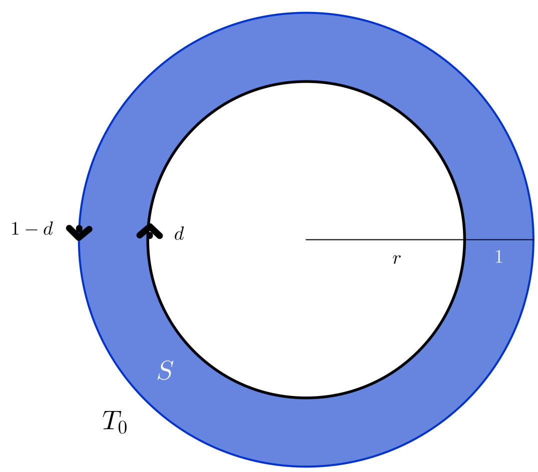


Figure 4.1

Proof. Note that

$$G(r,d) \geq F(r,d) \coloneqq 4\pi^2 \left[(1-d)^{2\alpha} + d^{2\alpha} r^2\right] + \frac{\pi d^\alpha [(1-r^2)]^2}{h}$$

since the cross terms of G are non-negative for every $0 \leq r \leq 1$ and $0 \leq d \leq 1$. Also note that $G(\sqrt{1-2h}, 1) = F(\sqrt{1-2h}, 1)$; thus, if F is minimized at $(\sqrt{1-2h}, 1)$, so must G.

The minimum of F must occur when $r = 0, 1$ or $\sqrt{1-2h}$ since

$$\frac{\partial F}{\partial r}(r,d) = 8\pi^2 d^{2\alpha} r - \frac{4\pi^2 d^{2\alpha} r(1-r^2)}{h} = 0;$$

$$r(2h - 1 + r^2) = 0.$$

So, we only need to consider $F(0,d), F(1,d)$ and $F(\sqrt{1-2h}, d)$.

First, we will show that the minimum for $F(\sqrt{1-2h}, d)$ occurs at $d = 1$. Note that

$$F(\sqrt{1-2h}, d) = 4\pi^2[(1-d)^{2\alpha} + (1-h)d^{2\alpha}],$$

and that $F((\sqrt{1-2h}, 0)) = 4\pi^2$ and $F((\sqrt{1-2h}, 1)) = 4\pi^2(1-h)$. So, we have

$$\frac{\partial F}{\partial d}((\sqrt{1-2h}, d)) = 4\pi^2(2\alpha)[-(1-d)^{2\alpha-1} + (1-h)d^{2\alpha-1}].$$

Thus, the critical point occurs at $d = (\sqrt[2\alpha-1]{1-h} + 1)^{-1}$. However,

$$F\left(\sqrt{1-2h}, (\sqrt[2\alpha-1]{1-h}+1)^{-1}\right) = 4\pi^2\left[\left(1 - (\sqrt[2\alpha-1]{1-h}+1)^{-1}\right)^{2\alpha} + \frac{1-h}{\left(\sqrt[2\alpha-1]{1-h}+1\right)^{2\alpha}}\right],$$

$$\geq 4\pi^2[1-h].$$

So, we must have that the minimum for $F(\sqrt{1-2h}, d)$ occurs at $d = 1$.

One may check that the above is indeed the global minimum by comparing it to the minima of $F((0,d))$ and $F((1,d))$ found through similar arguments. $\square$

Lemma 4.5. *The radius of T_t is given by $r(t) = \sqrt{r_0^2 - 2t}$.*

Proof. Let r_{j-1} be the radius of T^h_{j-1}. Then, the radius r_j of T^h_j minimizes

$$g(r) = [2\pi r]^2 + [\pi(r_{j-1}^2 - r^2)]^2.$$

So,

$$0 = g'(r_j) = 8\pi^2 r_j - \frac{4\pi r_j(r_{j-1}^2 - r_j^2)}{h};$$

$$r_{j-1}^2 - r_j^2 = 2h.$$

Take the sum of both sides from $j = 1$ up to ℓ:

$$r_0^2 - r_\ell^2 = \sum_{j=1}^{\ell} 2h;$$

For each h, choose ℓ so that $\ell h \leq t < (\ell + 1)h$ and let $h \to 0$ to obtain

$$r_0^2 - r(t)^2 = \int_0^t 2ds,$$

where $r(t)$ is the radius of T_t. Then, $r(t)$ must be differentiable and satisfy $r(0) = r_0$ and

$$\frac{dr}{dt} = -\frac{1}{r},$$

which has a solution:

$$r(t) = \sqrt{r_0^2 - 2t}.$$

$\square$

Remark. Note that Lemma 4.4 is false when $\alpha \in (1/2, 1]$ since density $d = (\sqrt[2\alpha-1]{1-h}+1)^{-1}$ actually minimizes. Hence, in this case, the flow "smears out" the initial circle so that it is difficult to analytically investigate.

While the above example only works for $\alpha \in (0, 1/2]$, we could have constructed a different flow having similar results for all $\alpha \in (0, 1]$. One would use the functional

$$G_j(T) = G_j(T^h_{j-1}, T, h) \coloneqq \mathbf{M}^\alpha(T) + \frac{\mathbf{Fill}^\alpha_K(T^h_{j-1} - T))}{\sqrt{h}}$$

when constructing the discrete sequences instead of Equation (3) in Chapter 3.

CHAPTER 5

Comparing the Constructed $\mathbf{M}^{\alpha}$ Mass Reducing Flow to the Mean Curvature Flow

We should address the natural question of how the $\mathbf{M}^{\alpha}$ mass reducing flow compares to the most studied geometric flow: the mean curvature flow.

1. Mean curvature flow

The motivation behind the mean curvature flow is as follows: suppose you want to flow a closed surface in $\mathbb{R}^3$ to decrease its surface area. Then, you should create a flow based on the shape of the surface. To decrease its area as quickly as possible, you should move convex pieces inwardly and concave pieces outwardly. Moreover, shape should also influence the flow's speed since flatter pieces should move more slowly. Eventually, you want the closed surface to shrink to a point or points.

In this section, we summarize the main results of the mean curvature flow and the curve shortening flow. For additional sources, see [**CMP15, Whi02, Gra89**].

1.1. Surfaces. The mean curvature flow originated in models for material sciences. In the 1950s, von Neumann studied a special case of the mean curvature flow on soap foams with interfaces having constant mean curvature [**vN52**]. However, the general mean curvature flow equation is believed to have first been written by Mullins in his work describing the coarsening in metals with interfaces of non-constant mean curvature [**Mul56**].

Informally, the mean curvature flow is a partial differential equation for evolving a surface, and in fact, it is the negative gradient flow for surface area. It initially behaves similarly to the ordinary heat equation since it encourages the surface to become smoother through averaging in small-scale variations. Eventually, however, the solution may become singular, as informally described above.

To be more formal, let M be a closed hypersurface in $\mathbb{R}^{n+1}$ and let M_t be a one-parameter family of hypersurfaces with $M_0 = M$. The first variation of volume is

$$\frac{d}{dt}\mathrm{Vol}(M_t) = \int_{M_t} \langle \partial_t x, H\mathbf{n} \rangle,$$

with x the position vector, $\mathbf{n}$ the unit normal, and H the mean curvature scalar, i.e.

$$H := \mathrm{div}_M(\mathbf{n}) = \sum_{i=1}^{n} \langle \nabla_{e_i}\mathbf{n}, e_i \rangle,$$

where $\{e_i\}$ is the orthonormal basis for M. This is equivalent to H being the sum of the principal curvatures of M.

Then, the gradient of volume is

$$\nabla \mathrm{Vol} = -H\mathbf{n}.$$

So, the most efficient way to reduce volume is to choose the one-parameter family so that

$$\partial_t x = -\nabla \mathrm{Vol} = -H\mathbf{n}.$$

If M_t flows by the negative gradient for volume, we say that it evolves by the mean curvature flow; and

$$\frac{d}{dt}\mathrm{Vol}(M_t) = -\langle \nabla \mathrm{Vol}, \nabla \mathrm{Vol} \rangle = -\int_{M_t} H^2.$$

Here, we list the basic properties of the mean curvature flow:

(1) Surfaces become smoother for a short time.

(2) Surface area decreases.

(3) The flow is collision-free: disjoint surfaces remain disjoint and embedded surfaces remain embedded.

(4) Compact, convex surfaces become extinct in finite time. This property is known as Huisken's Theorem:

THEOREM 5.1. [**Hui84**] *For $n \geq 2$, an n-dimensional, compact, convex surface in $\mathbb{R}^{n+1}$ must shrink to a "round point;" i.e., if the flow is rescaled to keep the enclosed volume constant, then the resulting surface converges to a round sphere at the extinction time.*

REMARK. Since Huisken's proof shows that the principal curvatures at each point of the asymptotic surface are all equal (though a priori they may vary from point to point) and since this condition is vacuously true for $n = 1$, the proof does not apply to the case of curves.

1.2. Curves. Even though the curve shortening flow is simply the 1-dimensional case of the mean curvature flow, it still possesses interesting properties in its own right:

(1) Curves become smoother for a short time.

(2) Arc length decreases.

(3) The flow is collision-free.

(4) Closed curves become extinct in finite time.

Partial proof for Property 3. Given two initially disjoint curves C_1 and C_2, we may assume without loss of generality that C_2 lies completely within C_1. Suppose to the contrary that C_1 and C_2 eventually collide. Denote the first intersection time by t. At time t, they must touch tangentially as in Figure 5.1. The curvature of C_2 at this point of tangency must be greater than or equal to the curvature of C_1. Say the inequality is strict. Then, C_2 must shrink faster than C_1 at this point. But this implies that at some earlier time $t - \epsilon$, the curves crossed each other, contradicting the choice of t. $\square$

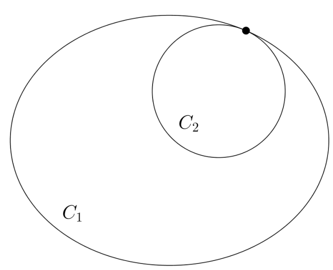


Figure 5.1

The main difference between the mean curvature flow and the curve shortening flow is Property 4. In 1986, Gage and Hamilton proved the analog to Theorem 5.1: the curve shortening flow shrinks convex curves to “round points” in finite time [**GH86**].

Theorem 5.2. *Under the curve shortening flow, a convex curve remains convex and shrinks to a point. Furthermore, it becomes asymptotically circular; i.e., if the* *flow is rescaled to keep the enclosed area constant, then the resulting curve converges to a round circle at the extinction time.*

EXAMPLE 5.3. Consider how the curve shortening flow acts on a circle of radius $r > 0$ in the plane. Since the circle has constant curvature, the flow will preserve the symmetry of the circle. By Theorem 5.2, the circle will shrink to a point. Explicitly solving the partial differential equation, we may find that an initial circle of radius r_0 will shrink to a circle of radius $\sqrt{r_0^2 - 2t}$ at time t. Additionally, it will become singular at time $t = \frac{r_0^2}{2}$.

A year later, Grayson extended Gage and Hamilton's theorem to *any* simple closed curve [**Gra87**]:

THEOREM 5.4. *Under the curve shortening flow, embedded closed curves become convex in finite time, and thus, they also shrink to "round points."*

At first glance, Theorem 5.4 may not be too surprising since it naturally stems from the collision-free Property 3 and the ability to enclose any embedded curve in a circle. However, Grayson's Theorem actually leads to some startling examples. For example, a snake-like coiled curve manages to unwind quickly enough to become convex and then shrink to a point. See Figure 5.2.

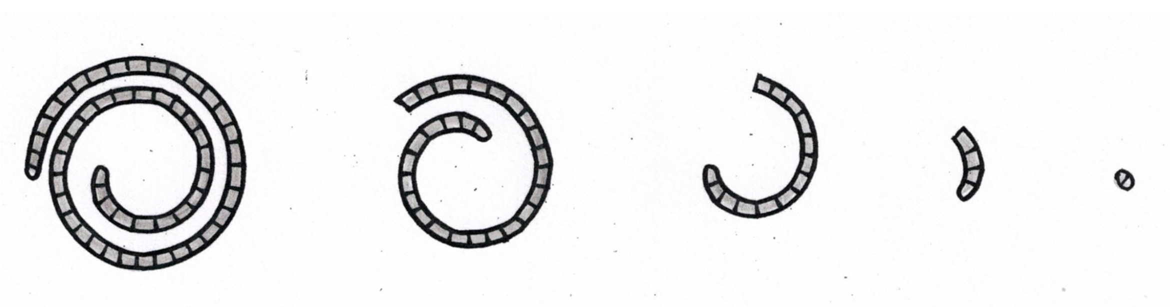


FIGURE 5.2

Theorem 5.4 is even more surprising because it does not hold for surfaces. Indeed, the convexity condition in Huiskin's Theorem 5.1 is necessary. To show this, Grayson gave an example of a dumbbell with a sufficiently long narrow bar. The bar will pinch off creating two separate spheres, which then continue to shrink to two points [**Gra89**]. See Figure 5.3.

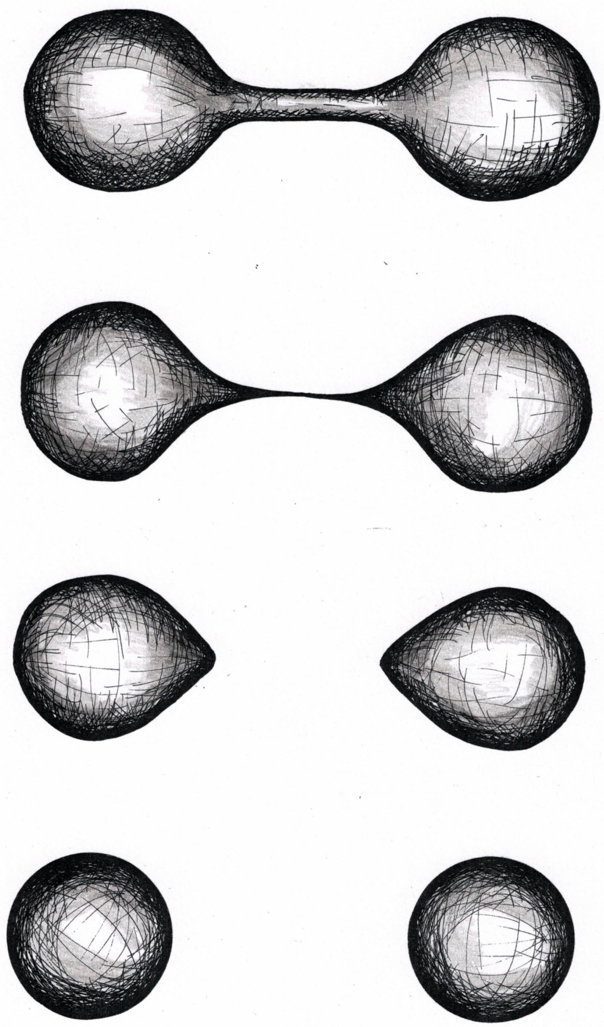


FIGURE 5.3

2. Similarities and differences

Unlike the mean curvature flow, the $\mathbf{M}^\alpha$ mass reducing flow for real-valued flat chains is a global construction, which consequently, often leads to difficulties in analyzing the evolutionary process. For example, the flow is non-unique, as noted in the following Subsection 2.1. However, as seen in the circle example (and will be seen in the second part of the thesis), one may use symmetry to analyze how the flow acts in specific cases.

The main advantage of this flow over the mean curvature flow is its capability of accommodating boundaries and real-valued weights. While Cheng's flow also has the boundary advantage [**Che93**], the $\mathbf{M}^\alpha$ mass reducing flow is particularly amenable to finding minimal cost transportation networks, as discussed later in Part 2.

We will now directly compare the two flows in the examples of a circle and a figure eight in the plane.

2.1. The circle revisited. The circle exemplifies an important difference between the two flows: non-uniqueness. In the rearrangement, we could have centered the annuli somewhere other than the origin as long as the support of S_ϵ^* is still contained in the convex hull of T_{j-1}^h. Because of this, the flow may evolve the initial chain to any point inside the circle for $\alpha \in (0, 1/2]$.

Despite this difference, the $\mathbf{M}^\alpha$ mass reducing flow for $\alpha \in (0, 1/2]$ shrinks circles at the same rate as the curve shortening flow (Example 5.3). In fact, we chose the functional in Equation (3) in Chapter 3 to ensure the achievement of this rate.

2.2. The figure eight.

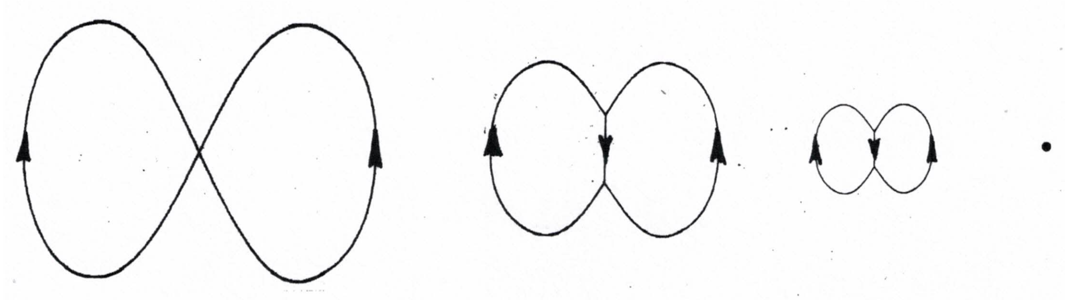
EXAMPLE 5.5. In the curve shortening flow, the figure eight parametrized as an embedded curve must flow to a single point as a consequence of Grayson's Theorem 5.4. However, this is not always the case in the $\mathbf{M}^\alpha$ mass reducing flow.

Consider flat chains in the shape of a figure eight with weight 1 and with orientations as shown in Figure 5.4. Locally near the crossing point, these chains look like transportation networks as in Example 7.5 in Chapter 7. Because of this, one may determine how the chain locally "zips up," either ensuring that the chain shrinks to one point or forcing the chain to break apart before shrinking to two distinct points. Whether the chain shrinks to one or two points is determined not by the orientation of the initial chain, but by the angles of the crossing. See Figure 5.4.

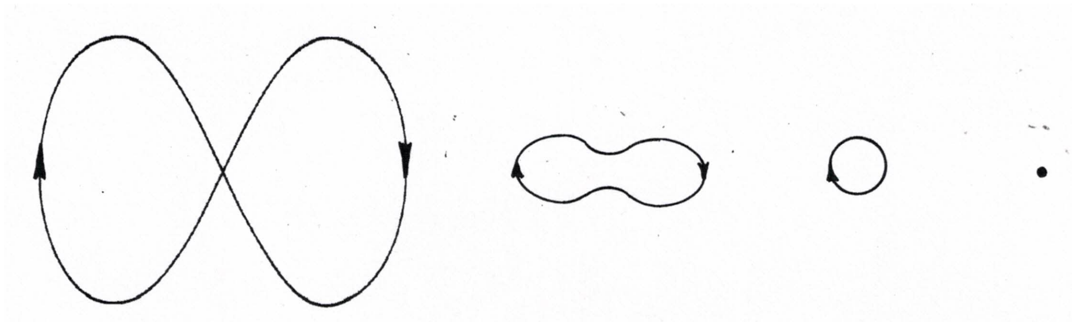
Thus, the $\mathbf{M}^\alpha$ mass reducing flow also differs from the curve shortening flow in the possible types of extinctions.

Despite the above example, we still might have an analogous result to Grayson's Theorem 5.4:

CONJECTURE 5.6. Rectifiable 1-cycles in the plane shrink to a countable number of points in finite time.



(A)



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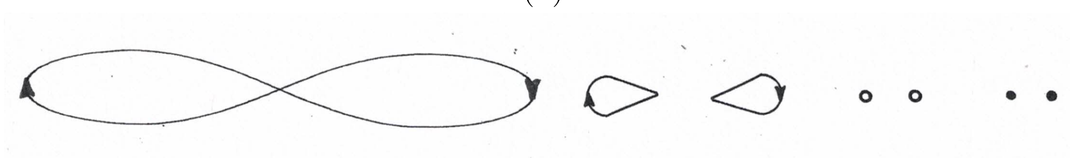
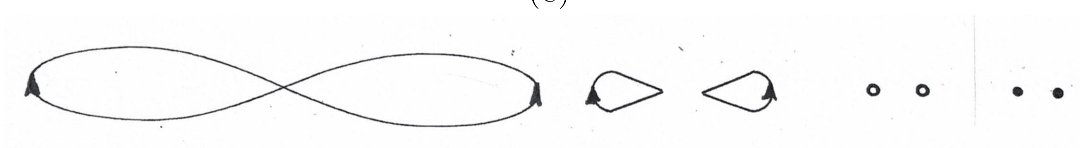


(C)



(D)

Figure 5.4. Notice that whether the figure eight evolves to one or two points is determined by the angles of the crossing.

Part 2

Applications to Transport Networks

CHAPTER 6

Transport Networks

Since we were motivated by optimal transport theory, we should analyze how the $\mathbf{M}^\alpha$ mass reducing flow acts on transportation networks. The second part of the thesis comprises this analysis.

The classical transportation model was first studied in 1781 by Monge, who analyzed how to efficiently move a pile of sand from one place to another [**Mon81**]. In 1942, Kantorovich formalized this model to accommodate transportation of positive measures [**Kan42**].

In the Monge-Kantorovich model, masses are transported linearly from supply to demand mass distributions, represented by positive measures on $\mathbb{R}^m$. Encoded into a positive measure π on $\mathbb{R}^m \times \mathbb{R}^m$, a transport scheme (also called a transference plan or transport plan) describes where to send each amount of supplied mass. To evaluate the efficiency of a transport scheme, one analyzes its cost given by

$$\int_{\mathbb{R}^m \times \mathbb{R}^m} c(x, y) d\pi(x, y),$$

where $c(x, y) : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^+$ represents the cost of transporting a unit mass from x to y. The minimization of this functional is called the Monge-Kantorovich problem. For more information, see [**Amb03, EG99, Eva97, GM96, Vil08**].

Notice that the model assumes each unit mass is transported along a straight line connecting its source to its destination. Because of this, the Monge-Kantorovich model cannot accommodate efficiencies of scale, and thus, cannot model ramified transportation networks. To overcome this limitation, Gilbert in 1967 used concave functions to incorporate efficiencies of scale into his model for communication networks [**Gil67**]. Gilbert associated a capacity $\lambda(e)$ to represent the amount of mass traveling along the edge e. Then, he required the cost per unit length of an edge with capacity $\lambda(e)$, denoted $f(\lambda)$, to be subadditive and increasing. That is, $f(\lambda_1)+f(\lambda_2) \geq f(\lambda_1+\lambda_2) \geq \max\{f(\lambda_1), f(\lambda_2)\}$. The simplest such cost is $f(\lambda) = \lambda^\alpha$ for $\alpha \in [0,1]$. In Gilbert's model, one then hopes to find a minimal cost transportation network between fixed supply and demand distributions. The subadditivity of f makes ramified structures advantageous.

In 2003, Gilbert's model was extended to a continuous framework, where the supply and the demand distributions may be arbitrary measures. In [**MMS03**], Maddalena, Solimini, and Morel gave a continuous formulation of the model in the case of a single supply source. They used Lipschitz paths to define fibers which transport unit masses, and then called a union of fibers a transport pattern, representing the network connecting the source to the demand distribution. Also in 2003, Xia gave a different continuous extension in which he modeled transportation networks as currents [**Xia03**]. Bernot, Caselles, and Morel gave an overview of these continuous extensions and bridge the two formulations in [**BCM09**].

In this chapter, we present an introduction to optimal transport theory as in Xia's work; however, we will use flat chains instead of currents to define transportation networks.

1. An overview: Definitions and main results

Let $\mathbf{a}, \mathbf{b}$ be two atomic measures of equal mass supported in $K \subset \mathbb{R}^m$; say

$$\mathbf{a} = \sum_{i=1}^{k} m_i \delta_{w_i} \text{ and } \mathbf{b} = \sum_{j=1}^{\ell} n_j \delta_{v_j},$$

where $\sum_{i=1}^{k} m_i = \sum_{j=1}^{\ell} n_j$. The $\{w_i\}_{i=1}^{k}$ are called sources and the $\{v_j\}_{j=1}^{\ell}$ are called sinks.

Definition 6.1. A *transportation network, or transport path, from* $\mathbf{a}$ *to* $\mathbf{b}$ is a weighted directed graph G whose vertex set $V(G)$, directed edge set $E(G)$, and weight function $\lambda : E(G) \to (0, +\infty)$ satisfy $\{w_i\}_{i=1}^{k} \cup \{v_j\}_{j=1}^{\ell} \subset V(G)$ and for any vertex $x \in V(G)$,

$$\sum_{e \in E(G), e^- = x} \lambda(e) = \sum_{e \in E(G), e^+ = x} \lambda(e) + \begin{cases} m_i & \text{if } x = w_i \text{ for some } i = 1, ..., k, \\ -n_j & \text{if } x = v_j \text{ for some } j = 1, ..., \ell, \\ 0 & \text{otherwise,} \end{cases} \tag{6}$$

where e^- and e^+ denote the starting and ending endpoints of edge $e \in E(G)$, respectively.

Let $Path(\mathbf{a}, \mathbf{b})$ denote the space of all transport paths from $\mathbf{a}$ to $\mathbf{b}$, which is always nonempty because the following "coned" network satisfies the above conditions:

$$G = \sum_{i=1}^{k} m_i [|(w_i, x)|] + \sum_{j=1}^{\ell} n_j [|(x, v_j)|],$$

where $x \in K$. Note that one may consider a transport path G from $\mathbf{a}$ to $\mathbf{b}$ as a polyhedral chain as in Definition 2.1. Thus, $Path(\mathbf{a},\mathbf{b}) \subset \mathcal{P}_1(K)$.

The balance equation (6) can be thought of as Kirchhoff's law or as mass being conserved at each vertex. In terms of polyhedral chains, this means that $\partial G = \mathbf{b} - \mathbf{a}$ when one views $\mathbf{a}, \mathbf{b}$ as 0-dimensional chains.

Definition 6.2. For any $\alpha \in [0,1]$ and for any $G \in Path(\mathbf{a},\mathbf{b})$, we may define the transportation cost of G to be its $\mathbf{M}^\alpha$ mass:

$$\mathbf{M}^\alpha(G) := \sum_{e \in E(G)} [\lambda(e)]^\alpha \mathcal{H}^1(e).$$

Lemma 6.3. [**Xia03**] *For any polyhedral 1-dimensional chain P, there exists a polyhedral 1-dimensional chain $\tilde{P}$ containing no cycles in its support with $\partial P = \partial \tilde{P}$ and $\mathbf{M}^\alpha(\tilde{P}) \leq \mathbf{M}^\alpha(P)$.*

Proof. Suppose that P contains some cycle L. For each edge $e \in L$, define

$$\beta(e) = \frac{\alpha \mathscr{H}^1(e)}{\lambda(e)^{1-\alpha}}.$$

Without loss of generality, fix an orientation of L, and set

$$L_1 := \sum \{[|e|] : \text{ edge } e \text{ has the same orientation as } L\},$$

and

$$L_2 := \sum \{[|e|] : \text{ edge } e \text{ has the opposite orientation as } L\}.$$

We may assume that

$$\sum_{e\in L_1}\beta(e)\le\sum_{e\in L_2}\beta(e),$$

since we may change the orientation of L if necessary.

Set $P' = P + B(L_1 - L_2)$ where $B \coloneqq \min\{\lambda(e) : e \in L_2\}$. Then, $\partial P' = \partial P$ and P' has one less cycle than P.

To complete the proof, consider the function on $[0, B]$ defined by

$$\begin{aligned} f(t) &\coloneqq \mathbf{M}^\alpha(P + t(L_1 - L_2)) - \mathbf{M}^\alpha(P), \\ &= \sum_{e\in L_1}\mathscr{H}^1(e)\big((\beta(e)+t)^\alpha - \beta(e)^\alpha\big) \\ &+ \sum_{e\in L_2}\mathscr{H}^1(e)\big((\beta(e)-t)^\alpha - \beta(e)^\alpha\big). \end{aligned}$$

Since $\alpha \in [0, 1]$, we must have $f''(t) \le 0$, $f'(t) \le f'(0) = \sum_{e\in L_1}\beta(e) - \sum_{e\in L_2}\beta(e) \le 0$, and $f(t) \le f(0) = 0$. Thus, we also have $\mathbf{M}^\alpha(P') \le \mathbf{M}^\alpha(P)$. We may repeat this process to eliminate all cycles in the considered polyhedral chain. $\square$

To define transport paths in more generality, let μ^+, μ^- be two Radon measures on $K \subset \mathbb{R}^m$ of equal total mass, which we may consider as 0-dimensional, real-valued flat chains of equal $\mathbf{M}^1$ mass.

Definition 6.4. Define a *transportation network or transport path from μ^+ to μ^-* to be a real-valued, 1-dimensional flat chain T (as in Definition 2.4) with boundary $\partial T = \mu^- - \mu^+$. Denote the space of all transport networks from μ^+ to μ^- by $Path(\mu^+, \mu^-)$.

Definition 6.5. For any $\alpha \in [0,1]$, also define the *cost of transporting μ^+ to μ^- via the transport path T* to be $\mathbf{M}^\alpha(T)$ as a real-valued flat chain.

Definition 6.6. An *optimal transportation network T from μ^+ to μ^-* is an element of $Path(\mu^+, \mu^-)$ of least $\mathbf{M}^\alpha$ mass.

By Lemma 6.3, you may assume that for any minimal transportation network T, there exists a sequence of polyhedral 1-chains $\{P_i\}$ containing no cycles with $\{P_i\}$ converging to T in the flat metric. Additionally, an optimal transport path will have its support, as a chain, contained in the convex hull of the union of the supports of μ^+ and μ^-:

Lemma 6.7. [**BCM09**] *An optimal transportation network $T \in Path(\mu^+, \mu^-)$ must be supported in the convex hull of the union of the supports of μ^+ and μ^-.*

Proof. Otherwise, we may use projection and decrease the $\mathbf{M}^\alpha$ mass by Lemma 2.13 □

In [**Xia03, Xia04, Xia11a**], Xia investigates existence and regularity questions for these generalized transportation networks. His results are summarized below.

Theorem 6.8. *There exists an optimal transport path from μ^+ to μ^- for any $\alpha \in (1 - \frac{1}{m}, 1]$.*

Using results from [**Whi99b**], Xia showed in [**Xia04, Xia11a**] some regularity theory. In particular, he proved that any optimal transportation network with finite $\mathbf{M}^\alpha$ cost is rectifiable. Moreover, he proved that an optimal transport path of finite cost indeed possesses nice regularity. The next two theorems comprise these results.

Theorem 6.9. [**Xia04**] *Any optimal transport path $T \in Path(\mu^+, \mu^-)$ with finite $\mathbf{M}^\alpha$ cost for $\alpha \in [0,1)$ is a rectifiable 1-chain with*

$$\mathbf{M}^\alpha(T) = \int_M \lambda(x)^\alpha d\mathcal{H}^1 < +\infty,$$

where M is an $\mathcal{H}^1$ measurable, 1-dimensional, rectifiable subset of K and λ is an $\mathcal{H}^1\big|_M$ integrable, positive function.

Theorem 6.10. [**Xia04**] *If $T \in Path(\mu^+, \mu^-)$ is an optimal transport path with finite $\mathbf{M}^\alpha$ cost, then there is an open neighborhood B_p of any point $p \in spt(T) \setminus spt(\mu^+ \cup \mu^-)$ such that $T \llcorner B_p$ is a cone consisting of finitely many line segments with suitable multiplicities.*

2. Construction of optimal transport networks

Despite the above existence and regularity theory, explicitly constructing optimal networks, even between finite atomic measures, has proven to be difficult. The difficulty lies in locating the bifurcation points that optimally take advantage of the cost benefits afforded by α. In this section, we present an overview of the existing literature attempting to construct optimal paths between atomic measures. We begin with a lemma giving angle constraints for the location of a bifurcation point in an optimal structure for two sources and one sink. Then, we present a construction from Gilbert's paper [**Gil67**] and Bernot's PhD thesis [**Ber05**] for an optimal transport path within a fixed topology, as presented in [**BCM09**].

2.1. Two sources; one sink. Let $\mu^+ = m_1\delta_{w_1} + m_2\delta_{w_2}$ and $\mu^- = (m_1+m_2)\delta_v$ for $m_1, m_2 \in \mathbb{R}$ and $w_1, w_2, v \in \mathbb{R}^m$.

LEMMA 6.11. *If w_1, w_2, v are aligned, then the optimal transport path from μ^+ to μ^- is simply supported in the line segment connecting the three points. Otherwise, the optimal transport path is supported in the triangle formed by the three points, allowing one to assume $w_1, w_2, v \in \mathbb{R}^2$. Moreover, the optimal network is a directed graph with either two or three edges.*

PROOF. By Lemma 6.7, we know that the support of the optimal path must be in the convex envelop of the three points. It follows from Lemma 6.3 that the optimal path must also be a directed graph with at most three edges. $\square$

From now on, assume $w_1, w_2, v \in \mathbb{R}^2$.

LEMMA 6.12. *If $G \in Path(\mu^+, \mu^-)$ is optimal and made up of three edges, then the interior vertex B of G must satisfy the following angle constraints:*

$$\cos(\theta_1) = \frac{k_1^{2\alpha} + 1 - k_2^{2\alpha}}{2k_1^\alpha}; \tag{7}$$

$$\cos(\theta_2) = \frac{k_2^{2\alpha} + 1 - k_1^{2\alpha}}{2k_2^\alpha}; \tag{8}$$

and

$$\cos(\theta_1 + \theta_2) = \frac{1 - k_1^{2\alpha} - k_2^{2\alpha}}{2k_1^\alpha k_2^\alpha}, \tag{9}$$

where $k_1 = \frac{m_1}{m_1+m_2}$, $k_2 = \frac{m_2}{m_1+m_2}$, and θ_1, θ_2 are as shown in Figure 6.1.

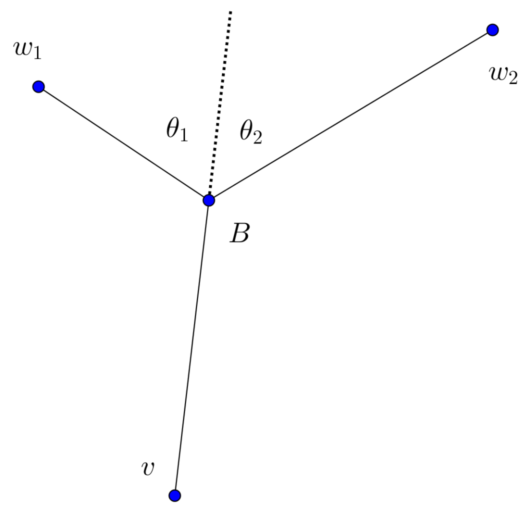


FIGURE 6.1. If G is optimal and has three edges as shown, then θ_1 and θ_2 must satisfy the angle constraints in Lemma 6.12.

PROOF. For $B \in \mathbb{R}^2 \setminus \{w_1, w_2, v\}$, let the path $G(B)$ represent the path with edges $[|w_1B|]$, $[|w_2B|]$, and $[|Bv|]$ with associated weights m_1, m_2, and $m_3 = m_1 + m_2$, respectively. Then, its cost is

$$\mathbf{M}^{\alpha}\left(G(B)\right) = m_1^{\alpha}|w_1B| + m_2^{\alpha}|w_2B| + m_3^{\alpha}|Bv|,$$

which is a differentiable function with respect to B on $\mathbb{R}^2 \setminus \{w_1, w_2, v\}$. If $G(B)$ is optimal, then its first derivative must be zero. Denote by $\mathbf{n_i} = \dfrac{\mathbf{Bw_i}}{||\mathbf{Bw_i}||}$ the unit vector from B to w_i for $i = 1, 2$, and similarly, denote by $\mathbf{n_3} = \dfrac{\mathbf{vB}}{||\mathbf{vB}||}$ the unit vector from v to B. Then, the first derivative vanishing is equivalent to the following balance equation:

$$m_1^{\alpha}\mathbf{n_1} + m_2^{\alpha}\mathbf{n_2} + m_3^{\alpha}\mathbf{n_3} = 0. \tag{10}$$

Denote by θ_i the angle between $\mathbf{n_i}$ and $\mathbf{n_3}$ as shown in Figure 6.1, and let $k_1 = \frac{m_1}{m_1+m_2}$ and $k_2 = \frac{m_2}{m_1+m_2}$. Taking the inner product of Equation (10) with each $\mathbf{n_i}$ yields the following three equations:

$$k_1^\alpha + k_2^\alpha \mathbf{n_1} \cdot \mathbf{n_2} = \cos(\theta_1);$$

$$k_1^\alpha \mathbf{n_1} \cdot \mathbf{n_2} + k_2^\alpha = \cos(\theta_2);$$

and

$$k_1^\alpha \cos(\theta_1) + k_2^\alpha \cos(\theta_2) = 1.$$

Using this system of equations, one finds Equations (7), (8), and (9). □

Remark. When $m_1 = m_2$, then $\theta_1 = \theta_2 = \arccos(2^{2\alpha-1} - 1)/2$. If $\alpha = 0.5$, then $\theta_1 + \theta_2 = \pi/2$, $\theta_1 = \sqrt{k_1}$ and $\theta_2 = \sqrt{k_2}$. In this case, the bifurcation point B will lie on circle with diameter $w_1 w_2$. If $\alpha = 0$, then $\theta_1 + \theta_2 = 2\pi/3$. That is, the bifurcation point is the Steiner point of the desired directed Steiner tree [**BZ15**].

The angle constraints in Lemma 6.12 help locate the optimal bifurcation point. However, one should note that the "Y" shaped network is not necessarily the optimal path, which instead may have an "L" or "V" shape. Below, we first locate the optimal bifurcation point in the "Y" structure, and then, completely classify the possible optimal structures for transport paths between two sources and one sink.

Lemma 6.13. [**BCM09**] *Given two points w_1, w_2 and an angle $\theta \leq \frac{\pi}{2}$, the set of points B such that the non-oriented angle $w_1 B w_2$ equals θ is the union of two circle arcs through w_1 and w_2 with radius $\frac{|w_1, w_2|}{2\sin(\theta)}$. These circle arcs are called equiangular arcs.*

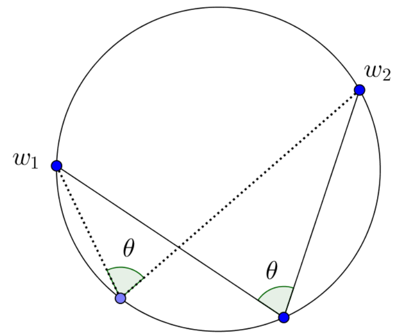


FIGURE 6.2. Equiangular circle for w_1 and w_2.

LEMMA 6.14. [**BCM09**] *Let $G \in Path(\mu^+, \mu^-)$ be an optimal transport path with three edges. Let $\mathscr{C}$ be the equiangular arc associated to w_1, w_2 in the same half plane as v, and let $\theta = \theta_1 + \theta_2$ as in Lemma 6.1. Let $\mathscr{C}'$ be the complementary circle arc of $\mathscr{C}$. There is a "pivot" point $w_{1,2} \in \mathscr{C}'$, not depending on v, such that the bifurcation point of G is the intersection of $w_{1,2}v$ with $\mathscr{C}$.*

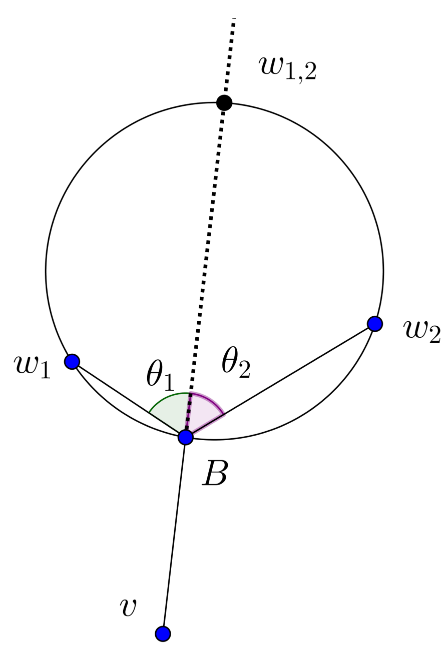


FIGURE 6.3. Pivot point $w_{1,2}$

THEOREM 6.15. [**BCM09**] *Let $G \in Path(\mu^+, \mu^-)$ be an optimal transport path with three edges and let $w_{1,2}$ be the "pivot" point obtained in Lemma 6.14. Then,*

$$\mathbf{M}^\alpha(G) = (m_1 + m_2)^\alpha |w_{1,2}v|.$$

Now, we can completely classify all possible tree structures for optimal transport paths from μ^+ to μ^-.

THEOREM 6.16. [**BCM09**] *Let $G \in Path(\mu^+, \mu^-)$ be an optimal transport path. Let $w_{1,2}$ be the associated "pivot" point to w_1, w_2 with weights m_1, m_2, respectively, in the half plane not containing v. Let $\mathscr{C}$ be the equiangular circle arc that is in the same half plane as v. There are four different zones for v and these zones determine the structure of G:*

- *If $w_{1,2}v \cap \mathscr{C} = \{B\}$ and $B \in [w_{1,2}v]$, then G has a "Y" structure. (Figure 6.4a)*
- *If $w_{1,2}v \cap \mathscr{C} = \{B\}$ and $B \notin [w_{1,2}v]$, then G has a "V" structure. (Figure 6.4b)*
- *If $w_{1,2}v \cap \mathscr{C} = \emptyset$, then G has an "L" structure. Then, the lengths $|w_1v|$ and $|w_2v|$ determine the structure:*
  - *If $|w_1v| < |w_2v|$, then G is made of the directed edges w_2w_1 and w_1v with weights m_2 and $m_1 + m_2$, respectively. (Figure 6.4c)*
  - *If $|w_2v| < |w_1v|$, then G is made of the directed edges w_1w_2 and w_2v with weights m_1 and $m_1 + m_2$, respectively. (Figure 6.4d)*

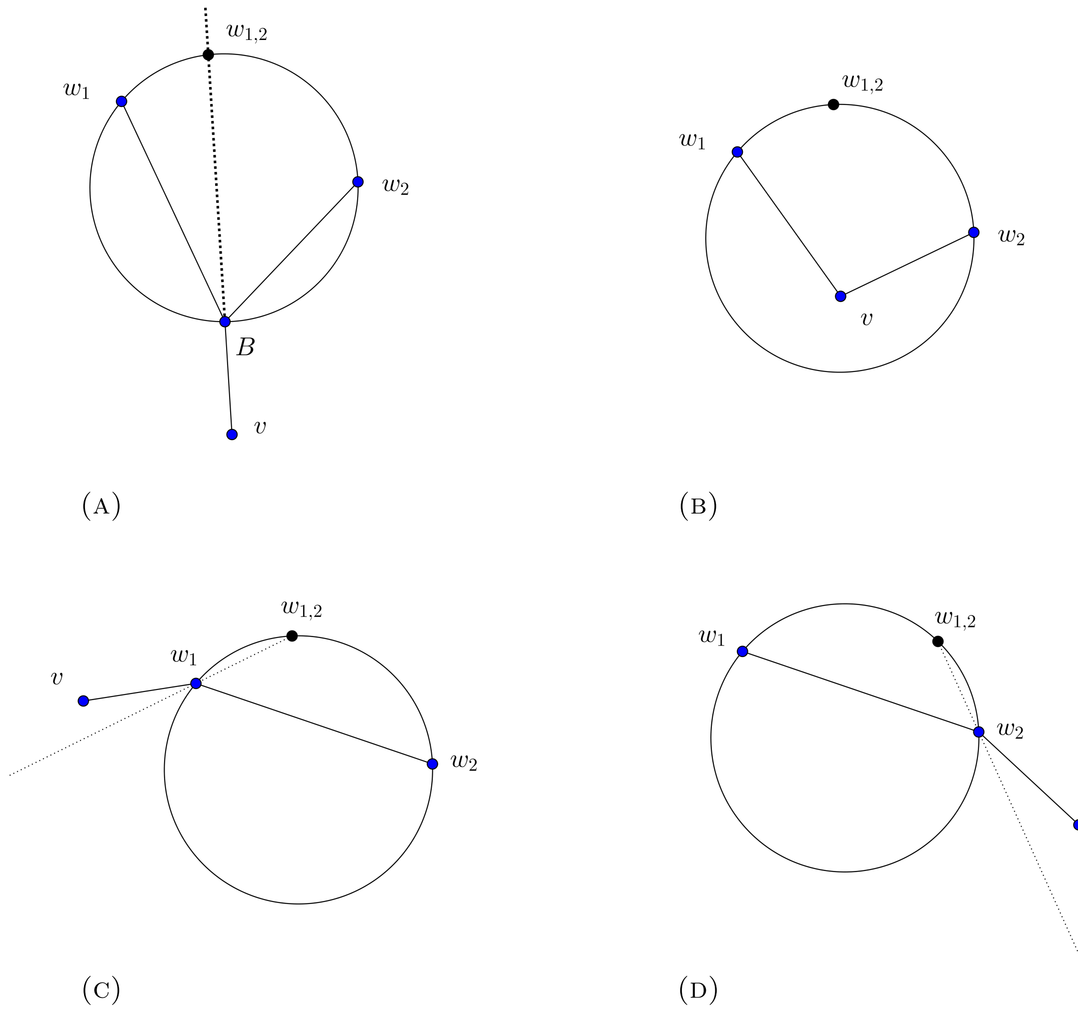


FIGURE 6.4. The different zones that determine the optimal structure

Here, we summarize the construction of the optimal transportation network from two sources to one sink:

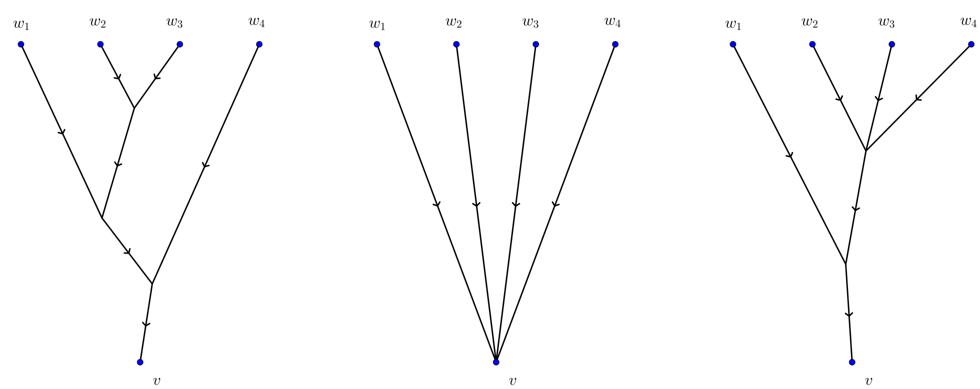
(1) Using the equation in Lemma 6.12, find the angle θ at an optimal bifurcation point.

(2) Draw the equiangular arc, the "pivot" point, and the line from the "pivot" point to v.

(3) Doing so, we obtain one of the four possible configurations described above which determine the structure of the optimal transport path.

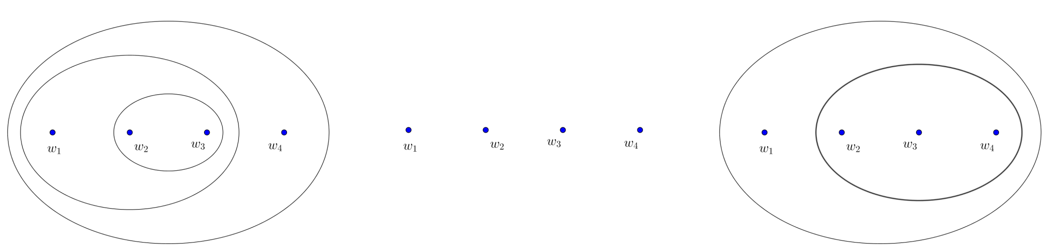
2.2. More sources. Bernot extended the above construction for more than two sources [**Ber05**]. However, his construction is limited to an optimal network within a fixed topology, defined below.

Definition 6.17. A *topology* $\mathscr{T}$ of a transport path between atomic measures is the topology of its underlying (undirected) tree. In other words, the topology of a transportation network describes its branching structure by encoding the information of how vertices are connected. One may use a nested graphical notation to represent the topology of a tree. (See Figure 6.6.) A curve enclosing multiple sources signifies that a bifurcation point connects these sources.



(A) (B) (C)

Figure 6.5. Examples of different topologies.



(A) (B) (C)

Figure 6.6. The encoded topologies of the transport paths in Figure 6.5.

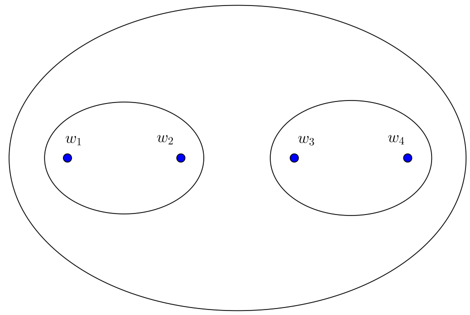


FIGURE 6.7. The topology for the construction example

DEFINITION 6.18. For transport paths between $\mu^+ = \sum_{i=1}^{k-1} m_i \delta_{w_i}$ and $\mu^- = (\sum_{i=1}^{k-1} m_i)\delta_v$, a *full topology* is a topology such that there are $2k-2$ distinct vertices, $2k-3$ distinct edges, and each interior vertex has degree three.

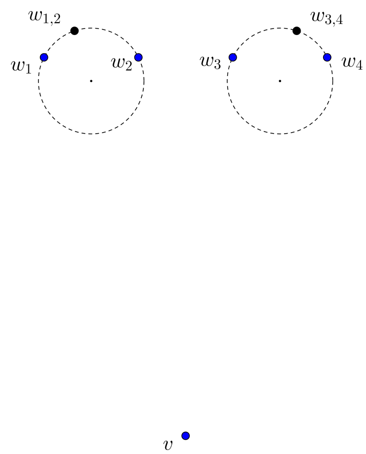
Now, we will give the construction for an optimal transport path within a fixed full topology. Since every subtree must be optimal, one may recursively apply the results for the two source case. Instead of giving the construction in full generality, we will outline a demonstrative example with four sources; see Figure 6.8:

EXAMPLE 6.19. Let $\mu^+ = \sum_{i=1}^{4} m_i \delta_{w_i}$ and $\mu^- = (\sum_{i=1}^{4} m_i)\delta_v$, and suppose the fixed full topology is the one encoded by Figure 6.7.

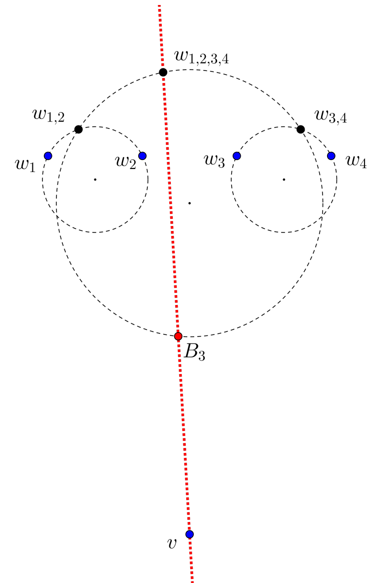
(1) Construct the pivot points $w_{1,2}$ and $w_{3,4}$ as in Lemma 6.14.
(2) The topology forces bifurcation points B_1 and B_2 to be grouped together. So, we must create the pivot point associated to $w_{1,2}$ and $w_{3,4}$ with their respective combined weights of $m_1 + m_2$ and $m_3 + m_4$. Call this pivot point $w_{1,2,3,4}$.

(3) We know that B_3 must be the intersection of the line $w_{1,2,3,4}v$ and the equiangular circle through points $w_{1,2}$,$w_{3,4}$, and $w_{1,2,3,4}$. This follows from Lemma 6.15 and the fact that the subtree consisting of edges $w_{1,2}B_3$, $w_{3,4}B_3$, and B_3v must be optimal.

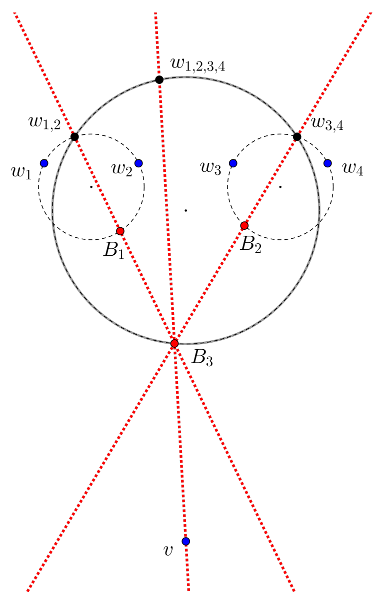
(4) After locating B_3, we may locate B_1 and B_2 via a similar technique. The location of B_1 is the intersection of the line $w_{1,2}B_3$ and the equiangular circle through points w_1, w_2, and $w_{1,2}$. And likewise, the location of B_2 is the intersection of the line $w_{3,4}B_3$ and the equiangular circle through points w_3, w_4, and $w_{3,4}$.



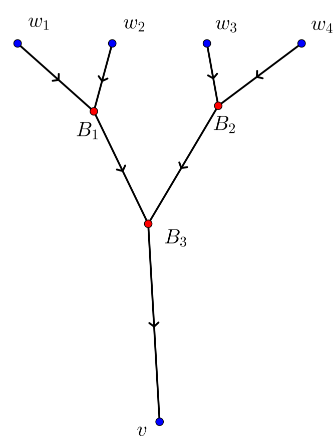
(A) First, construct pivot points $w_{1,2}$ and $w_{3,4}$.



(B) Then, construct pivot point $w_{1,2,3,4}$ using pivot points $w_{1,2}$ and $w_{3,4}$. B_3 must be the intersection of the line $w_{1,2,3,4}v$ and the circle through points $w_{1,2}$,$w_{3,4}$, and $w_{1,2,3,4}$.



(C) B_1 is the intersection of the line $w_{1,2}B_3$ and the circle through points w_1, w_2, and $w_{1,2}$. And likewise, B_2 is the intersection of the line $w_{3,4}B_3$ and the circle through points w_3, w_4, and $w_{3,4}$.



(D) Draw the resulting optimal transportation network.

FIGURE 6.8. The construction within the fixed full topology as shown in Figure 6.7

CHAPTER 7

The Flow Applied to Transport Networks

1. Zipping and unzipping

Despite having regularity results, we often only know the exact structure of optimal transportation networks for specific examples. This is due to computational difficulties. To find a global minimum, one would need to apply the Bernot construction to every possible topology, which is computationally unfeasible for a large number of sources/sinks. So, instead of solving this optimization problem outright, we created the $\mathbf{M}^\alpha$ mass reducing flow to evolve transportation networks. Given some initial transport path, decrease its cost by applying the flow, encouraging a ramified structure. The flow may naturally change the topology of the initial path through a "(un)zipping" process, creating or eliminating interior vertices.

Since we may approximate any chain by a polyhedral chain in the flat-α norm, let us restrict our attention in this chapter to transport paths as polyhedral chains.

2. Examples

2.1. Two sources; one sink. To more concretely see the zipping process of the flow, let $\alpha \in (0, 1]$, $m_1, m_2 \in \mathbb{R}$, and $A, B, C \in \mathbb{R}^2$. Suppose that the optimal path is "Y" shaped. Without loss of generality, set $C = (0, 0)$ and let the positive x-axis extend from C to $V = (x_V, 0)$ where V is the optimal bifurcation point for the "Y" shape; that is, the x-axis breaks the angle $\angle ACB$ into θ_1 and θ_2 in Lemma 6.1. Write $A = (x_A, y_A)$ and $B = (x_B, y_B)$.

Let us see how the flow zips up the initial path $T_0 \in Path(\mu^+, \mu^-)$ having the "V" structure, i.e.,

$$T_0 = m_1[|A, C|] + m_2[|B, C|].$$

Lemma 7.1. *One may assume that each T_j^h for every $h > 0$ is a "Y" shaped path with bifurcation point on the x-axis, and thus, so is each T_t for almost every $t > 0$.*

Proof. By Lemma 6.7, we may assume that each path in the discrete sequences is contained in the convex hull of A, B, and C.

Now fix $h > 0$. We will prove via induction that every polyhedral path in our discrete sequence is "Y" shaped and its interior vertex lies on the x-axis. Let

$$T_{j-1}^h = m_1[|A, V_{j-1}|] + m_2[|B, V_{j-1}|] + (m_1 + m_2)[|V_{j-1}, C|],$$

where $V_{j-1} = (x_{j-1}, 0)$. (For $j = 1$, $V_{j-1} = (0, 0)$ so that T_{j-1}^h is exactly T_0.)

Suppose T minimizes functional G_j and let $V = (x, y)$ be its interior vertex. By Lemma 2.13, we may use projection to decrease the $\mathbf{M}^\alpha$ mass (and also, the $\mathbf{Fill}_K^\alpha$ norm), and so, we may assume that V lies in the convex hull of V_{j-1}, A and B. Without loss of generality, assume that the edge $[|V, C|]$ intersects $[|A, V_{j-1}|]$ at $\widetilde{V} = (\tilde{x}, \tilde{y})$. See Figure 7.1.

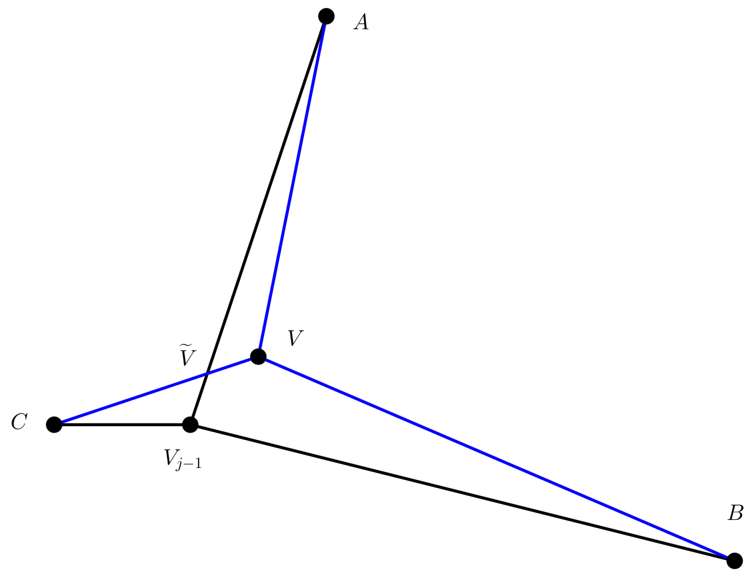


FIGURE 7.1

Then, we have

$$
\begin{aligned}
G(T) &= [\mathbf{M}^\alpha(T)]^2 + \frac{[\mathbf{Fill}^\alpha_K(T - T^h_j)]^2}{h};\\
G(V) &= \Big[m_1^\alpha |VA| + m_2^\alpha |VB| + (m_1+m_2)^\alpha |V| \Big]^2,\\
&\quad + \frac{1}{h}\Bigg[\frac{m_1^\alpha}{2} ||\overrightarrow{\widetilde{V}A} \times \overrightarrow{VA}|| + \frac{m_2^\alpha}{2} ||\overrightarrow{VB} \times \overrightarrow{V_{j-1}B}||,\\
&\quad + \frac{m_2^\alpha}{2} ||\overrightarrow{V_{j-1}\widetilde{V}} \times \overrightarrow{V_{j-1}V}|| + \frac{(m_1+m_2)^\alpha}{2} ||\overrightarrow{V_{j-1}} \times \overrightarrow{\widetilde{V}}|| \Bigg]^2.
\end{aligned}
$$

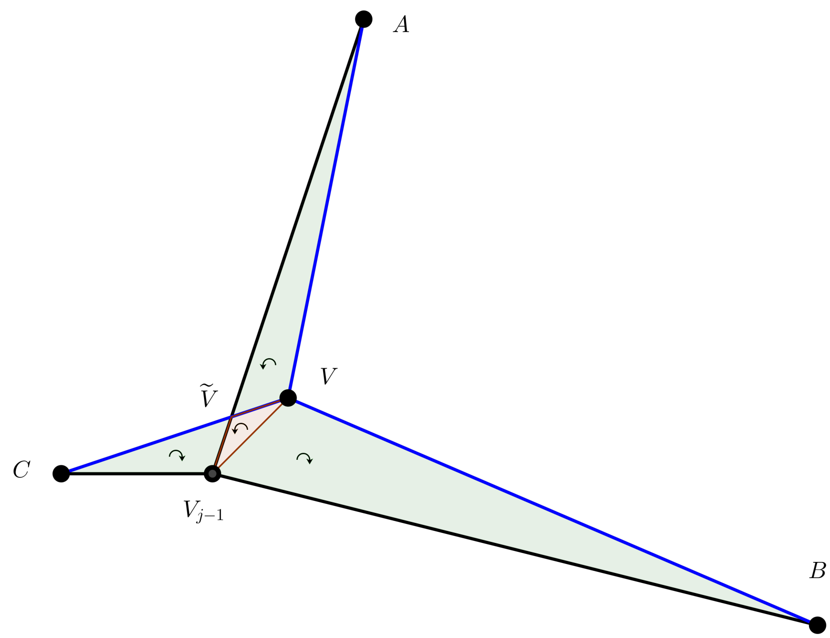


FIGURE 7.2

Writing the above equation in terms of (x, y) (including expressing $(\tilde{x}, \tilde{y})$ in terms of (x, y)), one can show that the above function is convex in y. And so, we have

$$G((x,0)) \leq \frac{G(x,y) + G(x,-y)}{2} = G((x,y)).$$

Hence, we may assume that T_j^h is "Y" shaped and that its interior vertex lies on the x-axis.

Moreover, the flow will stop zipping once the path has reached the optimal shape. One may see this by examining how any perturbation of the network with the optimal bifurcation point will increase the functional in Equation (3). $\square$

REMARK. Similarly, one may show that the flow unzips initial "Y" shaped paths with angles greater than that in Lemma 6.12 until the optimal bifurcation point is reached.

One may also compute the rate of zipping in specific examples by applying similar techniques as those in Chapter 4:

EXAMPLE 7.2. Let $\mu^+ = \delta_{(0,1)} + \delta_{(0,-1)}$ and $\mu^- = 2\delta_{(2,0)}$, and let $\alpha = 0.5$. Suppose the initial path is $T_0 = [|(0,1),(2,0)|] + [|(0,-1),(2,0)|]$. By Lemma 7.1, we know the flow zips up the path along the x-axis and ceases once an angle of $\pi/2$ is achieved. Here, we compute the rate of zipping.

Fix $h > 0$. Suppose $(x_{j-1}, 0)$ is the bifurcation point of T^h_{j-1}. If $(x_j, 0)$ denotes the bifurcation point of T^h_j, then it must be a critical point of the functional:

$$G_j(x) = \left[\sqrt{2}(2-x) + 2\sqrt{x^2+1}\right]^2 + \frac{\left[2\dfrac{(x_{j-1}-x)}{2}\right]^2}{h};$$

$$0 = G_j'(x_j) = 2\left[\sqrt{2}(2-x_j) + 2\sqrt{x_j^2+1}\right]\left[-\sqrt{2} + \frac{2x_j}{\sqrt{x_j^2+1}}\right] - \frac{2(x_{j-1}-x_j)}{h}.$$

Rearranging this and summing, we have

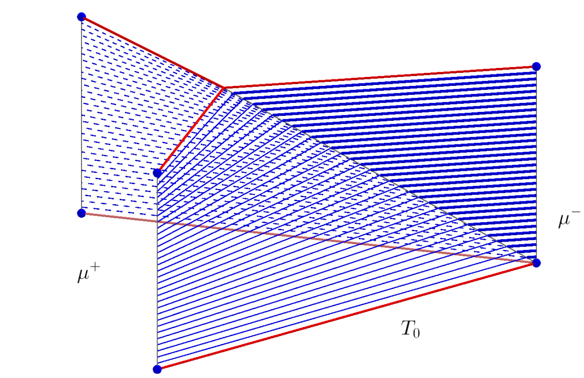
$$\sum_{j=1}^{\ell} x_j - x_{j-1} = -\sum_{j=1}^{\ell} h\left[\sqrt{2}(2-x_j) + 2\sqrt{x_j^2+1}\right]\left[-\sqrt{2} + \frac{2x_j}{\sqrt{x_j^2+1}}\right];$$

then, let $h \to 0$:

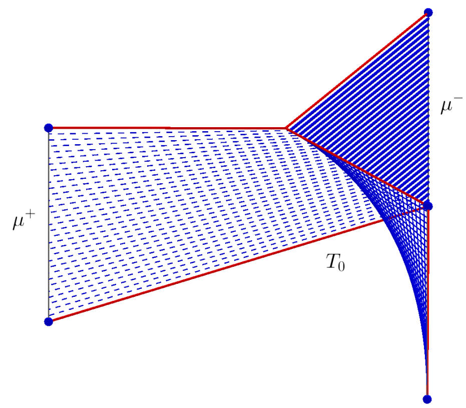
$$x(t) - x(0) = -\int_0^t \left[\sqrt{2}(2-x_j) + 2\sqrt{x_j^2+1}\right]\left[-\sqrt{2} + \frac{2x_j}{\sqrt{x_j^2+1}}\right] dt.$$

Hence, the rate of zipping is given by $x(t)$ which solves the following first order, non-linear ordinary differential equation: $x(0) = 2$ and

$$\frac{dx}{dt} = -\left[\sqrt{2}(2 - x_j) + 2\sqrt{x_j^2 + 1}\right]\left[-\sqrt{2} + \frac{2x_j}{\sqrt{x_j^2 + 1}}\right].$$



(A)



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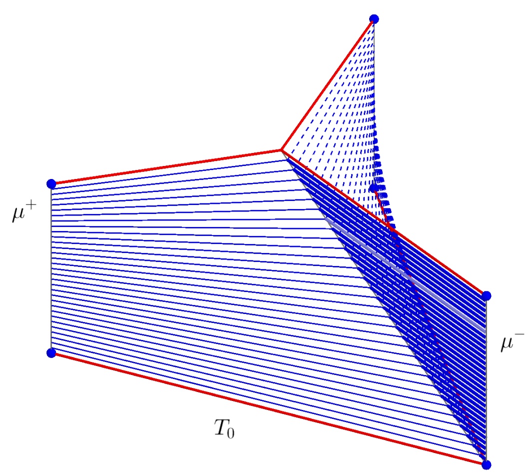


(C)

Figure 7.3. Here is a representation of the constructed, two dimensional flow S for two sources and one sink. (The rate of zipping will affect the curve where the top "fin" attaches.) We cut it off at the time when it finally reaches the optimal "Y" shape.

2.2. Two sources; two sinks. Suppose that $\mu^+ = m\delta_{(-w/2,\ell/2)} + m\delta_{(w/2,\ell/2)}$ and $\mu^- = m\delta_{(-w/2,-\ell/2)} + m\delta_{(w/2,-\ell/2)}$, for $m, w, \ell \in \mathbb{R}^+$. That is, the sources and sinks lie on the vertices of a rectangle of width w and length ℓ. We will now consider the flow acting on two different initial paths from μ^+ to μ^-: one with an "X" shape and the other with a "| |" shape.

First, we give a lemma classifying the possible optimal transportation networks:

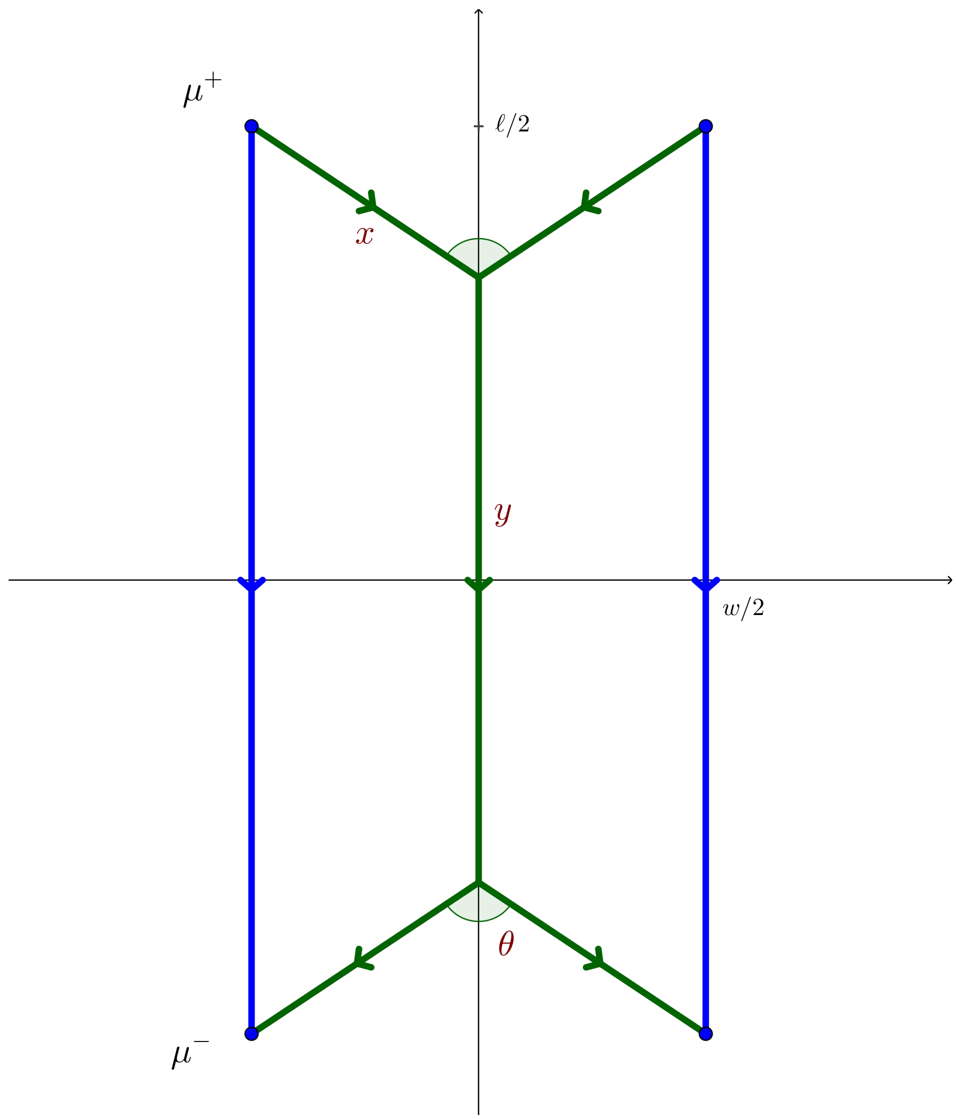


FIGURE 7.4. P_s is in blue and P_b is in green.

LEMMA 7.3. *There are two possible shapes for a minimal transport path between μ^+ and μ^-: a branching path (denoted P_b) or one composed of two straight edges (denoted P_s). For notation, let*

$$\begin{aligned} L &:= \frac{2w^2 - 4(2^{2\alpha-1}-1)\sin^2\left(\dfrac{\arccos\left(2^{2\alpha-1}-1\right)}{2}\right)}{(2-2^\alpha)w\sin\left(\dfrac{\arccos\left(2^{2\alpha-1}-1\right)}{2}\right)} \\ &= \frac{2(2w^2-(4-2^{2\alpha})(2^{2\alpha-1}-1))}{(2-2^\alpha)w\sqrt{4-2^{2\alpha}}}. \end{aligned}$$

If $\ell < L$, then P_s is optimal. If $\ell > L$, then P_b is optimal. And if $\ell = L$, then both are optimal since $\mathbf{M}^\alpha(P_s) = \mathbf{M}^\alpha(P_b)$.

PROOF. To show that P_b and P_s are the only possible optimal shapes for transport paths, one may use Lemma 6.3, Theorem 6.10, and Lemma 6.12. Note that $\mathbf{M}^\alpha(P_s) = 2m^\alpha\ell$ and

$$\mathbf{M}^\alpha(P_b) = 4m^\alpha x + (2m)^\alpha y,$$

where x and y are the lengths shown in Figure 7.4. Since we are transporting the same weight m, we know from Lemma 6.1 that $\theta_1 = \theta_2 = \arccos(2^{2\alpha-1}-1)/2$. Using trigonometry, we find

$$x = \frac{w}{2\sin\left(\dfrac{\arccos\left(2^{2\alpha-1}-1\right)}{2}\right)}$$

and

$$y = \ell - \frac{4\left(2^{2\alpha-1}-1\right)\sin\left(\dfrac{\arccos\left(2^{2\alpha-1}-1\right)}{2}\right)}{w}.$$

Thus, both paths are optimal when

$$\mathbf{M}^\alpha(P_s) = \mathbf{M}^\alpha(P_b);$$

$$2m^\alpha \ell = 4m^\alpha \left[\frac{w}{2\sin\left(\frac{\arccos\left(2^{2\alpha-1}-1\right)}{2}\right)} \right] + (2m)^\alpha \left[\ell - \frac{w\left(2^{2\alpha-1}-1\right)}{2\sin\left(\frac{\arccos\left(2^{2\alpha-1}-1\right)}{2}\right)} \right].$$

Solving, we find this happens when $\ell = L$. The other results follow. □

Example 7.4. Suppose T_0 is a transport path directly connecting each source to a sink in a "| |" shaped path. This network fails to evolve under the flow regardless of the value of ℓ. That is to say, for every $h > 0$, each T_j^h will be two parallel, directed line segments; and thus, T_t for almost every $t > 0$ will stay two parallel directed line segments. Hence, the flow finds a local minimum in this case.

Proof. One may see this by examining the discrete sequence's functional in Equation (3) in Chapter 3. With any perturbation of the previous path T_{j-1}^h, one would increase length of the directed segments and thus, the $\mathbf{M}^\alpha$ mass of the path, while simultaneously forcing a positive $\mathbf{Fill}_K^\alpha$ term. Hence, for every fixed $h > 0$, the discrete sequence must only be comprised of T_0. □

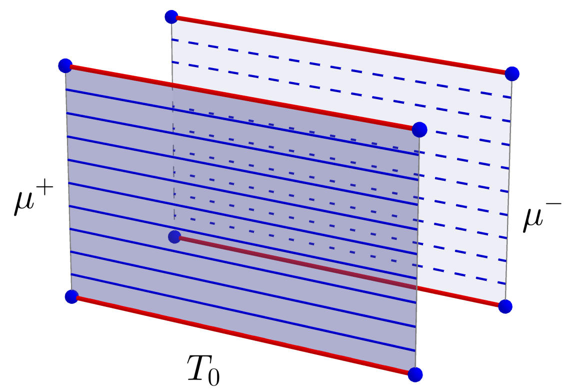


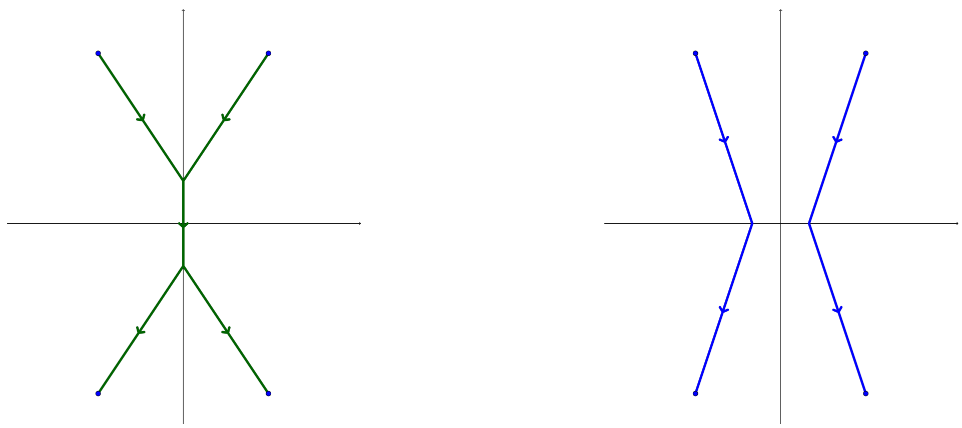
FIGURE 7.5. Here is a representation of the constructed two dimensional flow S for Example 7.4. We cut it off at some finite time.

EXAMPLE 7.5. Suppose we instead begin with an "X" shaped transport path T_0 from μ^+ to μ^-.

If $\ell = w$, then we may assume that every T_j^h for every $h > 0$ has a "$> <$" shape as in Figure 7.6b, and hence, so does every T_t for almost every $t > 0$. The path stabilizes under the flow once it reaches P_s, which is the optimal network by Lemma 7.3. In this case, the flow changes the topology of the network since we lose an interior vertex in the breaking process.

If $\ell = 4w$, then we may assume that each T_j^h for every $h > 0$ will have a branching structure as in Figure 7.6a; and thus, each T_t for almost every $t > 0$ will also have a branching structure. In this case, the path simultaneously zips up from both ends until reaching the angles of the optimal network P_b. Since we gain an interior vertex through the zipping process, the flow again changes the topology of the initial network.

Proof. Using similar convexity arguments as those used in Example 7.1, we may show that the paths must zip up along either axis due to the symmetry of μ^+ and μ^-. Zipping along the y-axis would result in an edge with associated weight $2m$, while zipping along the x-axis may be thought of as resulting in an edge with associated weight 0. See Figure 7.6.



(A) The branching structure when zipping up the y-axis.

(B) The "> <" structure when zipping up the x-axis.

Figure 7.6

Now, fix $h > 0$ and set $T_0^h = T_0$. Denote by T_x and T_y zipping up an $\epsilon > 0$ amount along the x-axis and y-axis, respectively. Then,

$$G_1(T_x) = \left[2m^\alpha\sqrt{\ell^2 + (w-\epsilon)^2}\right]^2 + \frac{\left[m^\alpha \ell\epsilon\right]^2}{4h},$$

and

$$G_1(T_y) = \left[(2m)^\alpha\epsilon + 2m^\alpha\sqrt{(\ell-\epsilon)^2 + (w)^2}\right]^2 + \frac{\left[m^\alpha w\epsilon\right]^2}{4h}.$$

If $\ell = w$, then $G_1(T_x) < G_1(T_y)$ for any $\epsilon > 0$. Hence, the initial path breaks apart and flows to P_s, as desired. See Figure 7.7.

On the other hand, if $\ell = 4w$, then $G_1(T_x) > G_1(T_y)$ for sufficiently small $\epsilon > 0$. Thus, the initial path must zip up along the y-axis and flow to P_b. See Figure 7.8. $\square$

Remark. While we only considered cases when zipping occurs, one may show similar results for paths that must unzip instead.

3. Open questions

Constructing optimal paths between atomic measures proves computationally difficult due to the vast number of possible topologies. Since the $\mathbf{M}^\alpha$ mass reducing flow is able to change the topology of the initial network, one hopes that the flow will overcome this difficulty, solving the transport optimization problem. Unfortunately, Example 7.4 shows that the flow generally finds merely local minima. However, Example 7.5 leads us to conjecture the following:

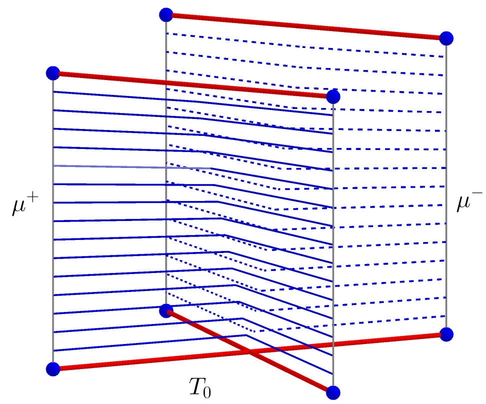
Conjecture 7.6. Let P_b, P_s, and L be as in Lemma 7.3. Suppose we begin with an "X" shaped transport path T_0 from μ^+ to μ^-.

If $\ell < L$, then the flow zips up T_0 along the x-axis, resulting in an edge with associated weight 0, until the network stabilizes after reaching the optimal path P_s.

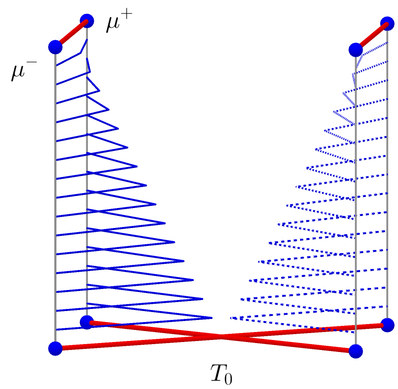
If $\ell > L$, then the flow zips up T_0 along the y-axis, resulting in an edge with associated weight $2m$, until the network stabilizes after reaching the optimal path P_b.

If $\ell = L$, then the flow is non-unique since it may zip up T_0 along either axis, flowing to either P_b or P_s.

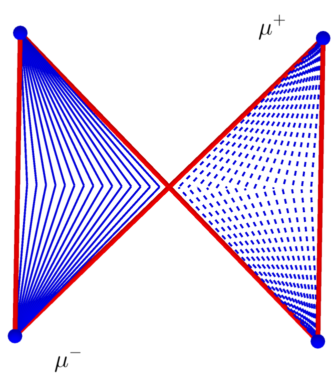
That is, we conjecture that, in any case, the initial "X" shaped path flows to the global minimal transportation network between μ^+ and μ^-.



(A)

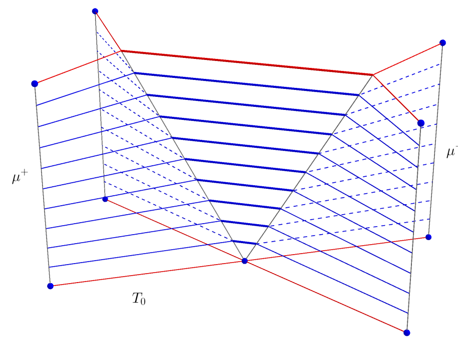


(B)

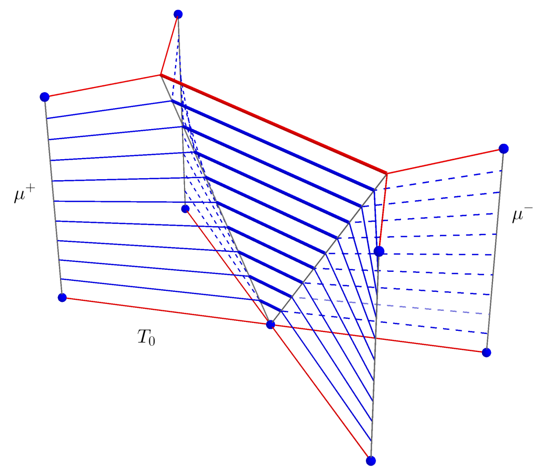


(C)

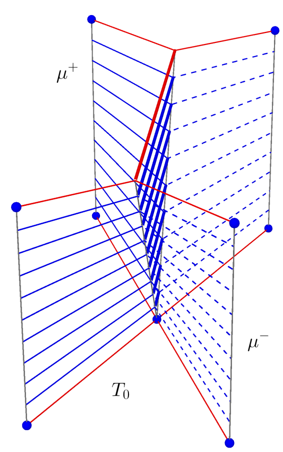
FIGURE 7.7. Here is a representation of the constructed, two dimensional flow S when $\ell = w$. (The rate of zipping will affect the shape.) We cut it off at the time when it achieves the optimal shape P_s.



(A)



(B)



(C)

FIGURE 7.8. Here is a representation of the constructed, two dimensional flow S when $\ell = 4w$. (The rate of zipping will affect the shape.) We cut it off at the time when it achieves the optimal shape P_b.

In fact, we may take the above conjecture further and hope to pick a portfolio of initial paths with at least one flowing to an optimal transportation network:

Conjecture 7.7. For atomic measures μ^+ and μ^-, an initial starting transport path composed of "coned" networks (having "hubs" for every connected component of an optimal path) will flow to an optimal transport network from μ^+ to μ^-.

Bibliography

[A.90] Schwarz H. A., *Gesammelte mathematische abhandlungen*, vol. 2, Springer-Verlag, 1890 (German).

[AJ66] F. J. Almgren Jr., *Plateau's problem*, Mathematics monograph series, W. A. Benjamin, Inc., 1966.

[Amb03] L. Ambrosio, *Lecture notes on optimal transport problems*, Springer, 2003.

[AT95] F. Almgren and J. Taylor, *Flat flow is motion by crystalline curvature for curves with crystalline energies*, Journal of Differential Geometry **42** (1995), no. 1, 1–22.

[ATW93] F. Almgren, J. Taylor, and L. Wang, *Curvature-driven flows: a variational approach*, SIAM Journal on Control and Optimization **31** (1993), no. 2, 387–438.

[BCM09] M. Bernot, V. Caselles, and J.-M. Morel, *Optimal transportation networks*, Lecture Notes in Mathematics 1955, Springer, 2009.

[Ber05] M. Bernot, *Irrigation and optimal transport*, Ph.D. thesis, Ph. D. Thesis, École Normale Supérieure de Cachan, 2005. Available at http://www. umpa. ens-lyon. fr/ mbernot, 2005.

[BLL74] H. J. Brascamp, E. H. Lieb, and J. M. Luttinger, *A general rearrangement inequality for multiple integrals*, Journal of functional analysis **17** (1974), no. 2, 227–237.

[Bra78] K. A. Brakke, *The motion of a surface by its mean curvature*, Princeton University Press, 1978.

[BZ15] Marcus Brazil and Martin Zachariasen, *Optimal interconnection trees in the plane*, Springer, 2015.

[Che93] X. Cheng, *A mass reducing flow for integral currents*, Indiana University mathematics journal **42** (1993), no. 2, 425–444.

[CMP15] T. Colding, W. Minicozzi, and E. Pedersen, *Mean curvature flow*, Bulletin of the American Mathematical Society **52** (2015), no. 2, 297–333.

[Dou39] J. Douglas, *Minimal surfaces of higher topological structure*, Annals of mathematics (1939), 205–298.

[DPH12] T. De Pauw and R. Hardt, *Rectifiable and flat g chains in a metric space*, American Journal of Mathematics **134** (2012), no. 1, 1–69.

[EG99] L. C Evans and W. Gangbo, *Differential equations methods for the monge-kantorovich mass transfer problem*, vol. 653, American Mathematical Society, 1999.

[ES91] L. Evans and J. Spruck, *Motion of level sets by mean curvature i*, Journal of Differential Geometry **33** (1991), no. 3, 635–681.

[ES92a] ______, *Motion of level sets by mean curvature. ii*, Transactions of the American mathematical society **330** (1992), no. 1, 321–332.

[ES92b] ______, *Motion of level sets by mean curvature iii*, The Journal of Geometric Analysis **2** (1992), no. 2, 121–150.

[ES95] ______, *Motion of level sets by mean curvature iv*, The Journal of Geometric Analysis **5** (1995), no. 1, 77–114.

[Eva97] L. C Evans, *Partial differential equations and monge-kantorovich mass transfer*, Current developments in mathematics **1997** (1997), no. 1, 65–126.

[Fed69] H. Federer, *Geometric measure theory*, Springer-Verlag, 1969.

[FF60] H. Federer and W. H. Fleming, *Normal and integral currents*, Annals of mathematics (1960), 458–520.

[Fle66] W. H. Fleming, *Flat chains over a finite coefficient group*, Transactions of the American mathematical society **121** (1966), no. 1, 160–186.

[GH86] M. Gage and R. Hamilton, *The heat equation shrinking convex plane curves*, Journal of Differential Geometry **23** (1986), no. 1, 69–96.

[Gil67] E. N. Gilbert, *Minimum cost communication networks*, Bell Labs Technical Journal **46** (1967), no. 9, 2209–2227.

[GM96] W. Gangbo and R. J. McCann, *The geometry of optimal transportation*, Acta Mathematica **177** (1996), no. 2, 113–161.

[Gra87] M. Grayson, *The heat equation shrinks embedded plane curves to round points*, Journal of Differential geometry **26** (1987), no. 2, 285–314.

[Gra89] ———, *Shortening embedded curves*, Annals of Mathematics **129** (1989), no. 1, 71–111.

[HHK04] J. Haga, K. Hoshino, and N. Kikuchi, *Construction of harmonic map flows through the method of discrete morse flows*, Computing and Visualization in Science **7** (2004), no. 1, 53–59.

[Hui84] G. Huisken, *Flow by mean curvature of convex surfaces into spheres*, Journal of Differential Geometry **20** (1984), 237–266.

[Kan42] L. V. Kantorovich, *On the translocation of masses*, Dokl. Akad. Nauk SSSR, vol. 37, 1942, pp. 199–201.

[Kaw85] B. Kawohl, *Rearrangements and convexity of level sets in pde*, Lecture Notes in Mathematics, vol. 1150, Springer-Verlag, 1985.

[Lei80] K. Leichtweiss, *Konvexe mengen*, Springer-Verlag, 1980 (German).

[MMS03] F. Maddalena, J.-M. Morel, and S. Solimini, *A variational model of irrigation patterns*, Interfaces and free boundaries **5** (2003), 391–415.

[Mon81] G. Monge, *Mémoire sur la théorie des déblais et de remblais*, Histoire de l'Académie Royale des Sciences de Paris (1781), 666–704 (French).

[Mor09] F. Morgan, *Geometric measure theory*, 4 ed., Academic Press, 2009.

[Mul56] W. Mullins, *Two-dimensional motion of idealized grain boundaries*, Journal of Applied Physics **27** (1956), no. 8, 900–904.

[Pla73] J. A. F. Plateau, *Statique expérimentale et théoreique des liquides soumis aux seules forces moléculaires*, vol. 2, Gauthier-Villars, 1873 (French).

[PS51] G. Pólya and G. Szegö, *Isoperimetric inequalities in mathematical physics*, no. 27, Princeton University Press, 1951.

[Rad30] T. Radó, *The problem of the least area and the problem of plateau*, Mathematische Zeitschrift **32** (1930), no. 1, 763–796.

[Rei60] E. R. Reifenberg, *Solution of the plateau problem form-dimensional surfaces of varying topological type*, Acta Mathematica **104** (1960), no. 1-2, 1–92.

[Vil08] Cédric Villani, *Optimal transport: old and new*, vol. 338, Springer Science & Business Media, 2008.

[vN52] J. von Neumann, *Metal interfaces*, American Society for Metals (1952), 108–110.

[Whi57] H. Whitney, *Geometric integration theory*, Princeton University Press, 1957.

[Whi99a] B. White, *The deformation theorem for flat chains*, Acta Mathematica **183** (1999), no. 2, 255–271.

[Whi99b] ______, *Rectifiability of flat chains*, Annals of mathematics **150** (1999), 165–184.

[Whi02] ______, *Evolution of curves and surfaces by mean curvature*, Proceedings of the ICM, 2002.

[Xia03] Q. Xia, *Optimal paths related to transport problems*, Communications in Contemporary Mathematics **5** (2003), no. 02, 251–279.

[Xia04] ______, *Interior regularity of optimal transport paths*, Calculus of Variations and Partial Differential Equations **20** (2004), no. 3, 283–299.

[Xia05] ______, *An application of optimal transport paths to urban transport networks*, Discrete and Continuous Dynmical Systems, Supplement Volume (2005), 904–910.

[Xia07] ______, *The formation of a tree leaf*, ESAIM: Control, Optimisation and Calculus of Variations **13** (2007), no. 2, 359–377.

[Xia11a] ______, *Boundary regularity of optimal transport paths*, Advances in Calculus of Variations **4** (2011), no. 2, 153–174.

[Xia11b] ______, *Ramified optimal transportation in geodesic metric spaces*, Advances in Calculus of Variations **4** (2011), no. 3, 277–307.

[Xia14] ______, *On landscape functions associated with transport paths*, Discrete and Continuous Dynamical Systems **34** (2014), no. 4, 1683–1700.

[Xia15] ______, *Motivations, ideas and applications of ramified optimal transportation*, ESAIM: Mathematical Modelling and Numerical Analysis **49** (2015), no. 6, 1791–1832.

[XV10] Q. Xia and A. Vershynina, *On the transport dimension of measures*, SIAM Journal on Mathematical Analysis **41** (2010), no. 6, 2407–2430.

[XX10] Q. Xia and S. Xu, *The exchange value embedded in a transport system*, Applied Mathematics and Optimization **62** (2010), no. 2, 229–252.